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4.9 **Thm:** Let $f, g : (\mathbb{X}, d) \to \mathbb{R}$ or $\mathbb{C}$ ($\|\cdot\|$ metric)
If f & g are continuous, then
$f+g$, cf, fg, f/g are continuous
(f/g requires $g \neq 0$ on $\mathbb{X}$)

4.10 **Thm:** Let $F = (f_1, f_2, \dots f_n) : (\mathbb{X}, d) \to (\mathbb{R}^n, \|\cdot\|)$
where $f_i : (\mathbb{X}, d) \to \mathbb{R}$.

F continuous on $\mathbb{X}$ $\Longleftrightarrow$ $\forall i$ $f_i : (\mathbb{X}, d) \to \mathbb{R}$ continuous

Proof: ($\Rightarrow$:) Fix $x \in \mathbb{X}$.

$$|f_i(x) - f_i(y)| \leq |F(x) - F(y)|$$

F cont at x: $\forall \varepsilon > 0\ \exists \delta > 0\ d(x, y) < \delta \Rightarrow \|F(x) - F(y)\| < \varepsilon$

$$|f_i(x) - f_i(y)| \leq \|F(x) - F(y)\| < \varepsilon$$

$\Leftarrow$: Fix x: $\forall \varepsilon > 0\ \exists \delta_i > 0\ d(x, y) < \delta_i \Rightarrow |f_i(x) - f_i(y)| < \frac{\varepsilon}{\sqrt{n}}$

Let $\delta = \min(\delta_1, \delta_2, \dots, \delta_n) > 0$

$\forall x, y$ $d(x, y) < \delta \leq \delta_i$

$$\|f(x) - f(y)\| = \left(\sum_{i=1}^{n} |f_i(x) - f_i(y)|^2 \right)^{\frac{1}{2}} < \left(\sum_{i=1}^{n} \left(\frac{\varepsilon}{\sqrt{n}}\right)^2 \right)^{1/2} = \varepsilon$$

Ex ① $f(x): \mathbb{R}^k \to \mathbb{R}^k$

$f(\vec{x}) = \vec{x}$ continuous, why?

$\forall\, U$ open $\subseteq \mathbb{R}^k$, $f^{-1}(U) = U$ is open.

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② $f: \mathbb{R}^k \to \mathbb{R}^k$

$f(x) = p_0$ (constant) continuous, why?

$$\forall\, U_{\text{open}} \subseteq \mathbb{R}^k \qquad f^{-1}(U) = \begin{cases} \phi & \text{if } p_0 \notin U \\ \mathbb{R}^k & \text{if } p_0 \in U \end{cases}$$

$\phi, \mathbb{R}^k$ are open in $\mathbb{R}^k$.

③ $f: \mathbb{R}^k \to \mathbb{R}$

$f_i(x_1, x_2 \ldots x_n) = x_i$ continuous

by Thm 4.10 & Ex ①

④ $f: \mathbb{R}^k \to \mathbb{R}$

$f(x_1, x_2, \ldots x_n) = \sum c_p \, x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ continuous

by Ex 3, Thm 4.9

Ex

$f(x,y,z) = x^2y^3 + xyz + 11xy^7z^6$

Ex $f(x) = \frac{1}{x}$: $\mathbb{R} - \{0\}$

Continuous at each $x \neq 0$
$\Rightarrow$ " on $\mathbb{R} - \{0\}$

$$g(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

not continuous on $\mathbb{R}$, regardless of what a is.

<u>Defn</u> A function $f : X \to \mathbb{R}^k$ is called bounded if $\exists M$ s.t.

$$\forall x \in X, \ |f(x)| \leq M.$$

<u>Recall</u> Thm: If $f : (X, d_X) \to (Y, d_Y)$ is continuous and X is compact, then $f(X)$ is compact.

<u>Corollary:</u> If $f : (X, d_X) \to (\mathbb{R}^k, \|\cdot\|)$ continuous, then:

X compact $\Longrightarrow$ $f(X)$ compact in $\mathbb{R}^k$
$\Longrightarrow$ $f(X)$ is closed & bounded
$\Longrightarrow$ f is a bounded function.

Thm (Extreme Value Thm)
Let $f:(X,d)\to\mathbb{R}$. If f is continuous and X is compact, then

$\exists\, p,q\in X$ s.t.

$\forall x\in X$ $\quad -\infty<\min f=f(p)\le f(x)\le f(q)=\max f<\infty$

In summary: A continuous real valued function on a compact metric space must attain its maximum and minimum values.

Proof: $\left.\begin{array}{l} X \text{ compact} \\ f \text{ continuous}\end{array}\right\} \Longrightarrow f(X)\subseteq\mathbb{R}$
$f(X)$ is compact.

$f(X)$ is closed and bounded.

$M=\sup f(X)$ exists by LUB property

$m=\inf f(X)$ " " LUB/GLB prop.

$m, M\in f(X)$ by Thm 2.28 $\quad \overline{f(X)}=f(X)$
since $f(X)$ is closed

$m\in f(X)\Rightarrow \exists\, p\in X\quad f(p)=m$
$M\in f(X)\Rightarrow \exists\, q\in X\quad f(q)=M$.

$\forall x\in X$
$-\infty<m=\min f(X)\le f(x)\le\max f(X)=M<\infty$.