March 19, 2018

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<u>Thm:</u> Let $f:(X, d_X) \to (Y, d_Y)$.

$f$ is continuous on $X$ $\iff$ $\forall U$ open in $Y$, $f^{-1}(U)$ is open in $X$.

<u>Proof:</u> ($\Rightarrow$:) done (Look at the notes from 3/9/18)

($\Leftarrow$:)

<u>Assume</u>: $\forall U$ open $\subseteq Y$, $f^{-1}(U)$ is open in $X$

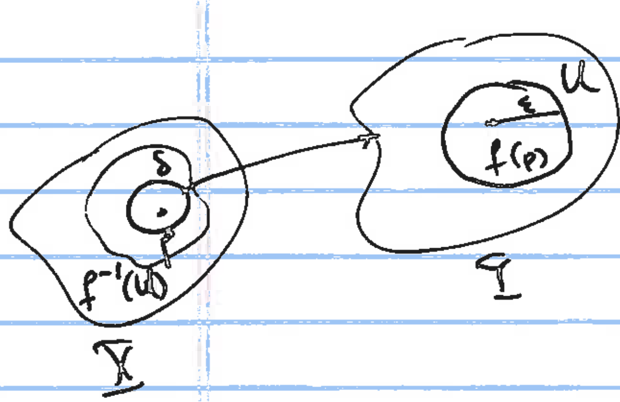
<u>To Show</u>:

$$\forall p \in X\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in X\ d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$$

Let $p \in X$ be given.

Let $\varepsilon > 0$ be given

Let $U = N_\varepsilon(f(p))$ open




Hypothesis $\Rightarrow$ $f^{-1}(U)$ is open.

$p \in f^{-1}(U)$ ($\Leftarrow f(p) \in U$)

$\exists \delta > 0$ s.t. $N_\delta(p) \subseteq f^{-1}(U)$

Let $x \in X$ s.t. $d_X(x,p) < \delta$

$x \in N_\delta(p) \subseteq f^{-1}(U)$

$x \in f^{-1}(U)$

$f(x) \in U = N_\varepsilon(f(p))$

$d_Y(f(x), f(p)) < \varepsilon$ #.

* **Corollary:** $f : (X, d_X) \to (Y, d_Y)$, a function between metric spaces. Then

$f$ continuous $\iff \forall C$ closed $\subseteq Y$, $f^{-1}(C)$ is closed in $X$.

**Lemma:** For any function $f : X \longrightarrow Y$

$$\forall C \subseteq Y \qquad X - f^{-1}(C) = f^{-1}(Y - C)$$

**Proof of Lemma** $\quad x \in f^{-1}(C) \iff f(x) \in C \quad$ Defn

$$x \notin f^{-1}(C) \iff f(x) \notin C$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$x \in X - f^{-1}(C) \qquad\qquad f(x) \in Y - C$$

$$\Updownarrow$$

$$x \in f^{-1}(Y - C)$$

**Proof of Corollary (*)** $(\Rightarrow:)$ [Hypothesis: $f$ continuous $(X, d_X) \to (Y, d_Y)$

Let $C$ be closed. $\subseteq Y$

$Y - C$ is open

$f^{-1}(Y-C)$ is open, since $f$ continuous

$X - f^{-1}(C) = f^{-1}(Y - C)$ open.

$(\Leftarrow$: HW$)$

$f^{-1}(C)$ is closed in $X$.

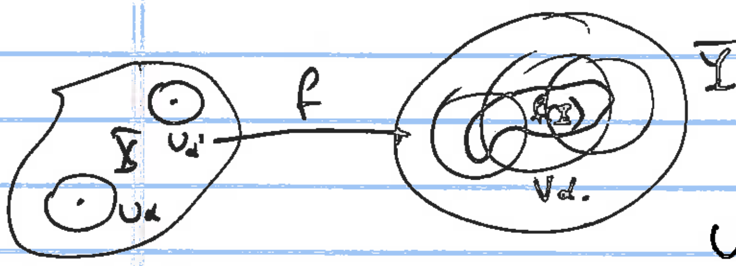
# CONTINUITY & COMPACTNESS

*** **Theorem** Let $f : (X, d_X) \to (Y, d_Y)$ be continuous.
Then:

$$X \text{ is compact} \implies f(X) \text{ is compact.}$$

**Proof:**

Let $\{V_\alpha \mid \alpha \in A\}$ be an open cover of $f(X)$




$$f(X) \subseteq \bigcup_{\alpha \in A} V_\alpha$$

$V_\alpha$ open in $Y$ $\forall \alpha \in A$

$U_\alpha = f^{-1}(V_\alpha)$ open in $X$ $\forall \alpha \in A$.

$$\forall x \in X \quad f(x) \in f(X) \subseteq \bigcup_{\alpha \in A} V_\alpha$$

$$\exists \alpha_0 \quad f(x) \in V_{\alpha_0}$$

$$x \in f^{-1}(V_{\alpha_0}) = U_{\alpha_0}.$$

Hence $X \subseteq \bigcup_{\alpha \in A} U_\alpha$, [ $\{U_\alpha \mid \alpha \in A\}$ is an open cover of $X$ ]

$X$ compact $\implies \exists \alpha_1, \alpha_2 \ldots \alpha_\ell$ s.t. $X \subseteq \bigcup_{i=1}^{\ell} U_{\alpha_i}$

Next: Want to establish $f(X) \subseteq \bigcup_{i=1}^{\ell} V_{\alpha_i}$

$$\mathbb{X} \subseteq \bigcup_{i=1}^{\ell} U_{\alpha_i}$$

$$f(\mathbb{X}) \subseteq f\left(\bigcup_{i=1}^{\ell} U_{\alpha_i}\right) = \bigcup_{i=1}^{\ell} f(U_{\alpha_i})$$

$$= \bigcup_{i=1}^{\ell} f(f^{-1}(V_{\alpha_i}))$$

$$\subseteq \bigcup_{i=1}^{\ell} V_{\alpha_i}$$

$f(\mathbb{X})$ is covered by a finite subcover of $\{V_\alpha \mid \alpha \in A\}$

$f(\mathbb{X})$ compact, since every open cover has a finite subcover. #

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We used the following:

Recall from 3770:

For any function $f : \mathbb{X} \to \mathbb{Y}$

i) $f(A) \cup f(B) = f(A \cup B) \quad \forall A, B \subseteq \mathbb{X}$

ii) $f(A) \cap f(B) \supseteq f(A \cap B) \quad \forall A, B \subseteq \mathbb{X}$

iii) $f(f^{-1}(C)) \subseteq C \quad \forall C \subseteq Y.$

<u>Also</u> $\left(\forall A, B \subseteq \mathbb{X} \quad f(A) \cap f(B) = f(A \cap B)\right) \iff f$ is 1-1

$\left(\forall C \subseteq \mathbb{Y} \quad f(f^{-1}(C)) = C\right) \iff f$ is onto.

**CAUTION** Let $f: X \to Y$ be continuous,

$A \subseteq X$,

$B \subseteq Y$.

| | |
|---|---|
| ① $A$ open $\not\Rightarrow f(A)$ open | Thm $B$ open $\Rightarrow f^{-1}(B)$ open. |
| ② $A$ closed $\not\Rightarrow f(A)$ closed | Thm $B$ closed $\Rightarrow f^{-1}(B)$ closed. |
| Thm $A$ compact $\Rightarrow f(A)$ compact | ③ $B$ compact $\not\Rightarrow f^{-1}(B)$ compact |

**Counterexamples:**

① $f(x) \equiv 1 : \mathbb{R} \to \mathbb{R}$ standard metric

$\mathbb{R}$ open

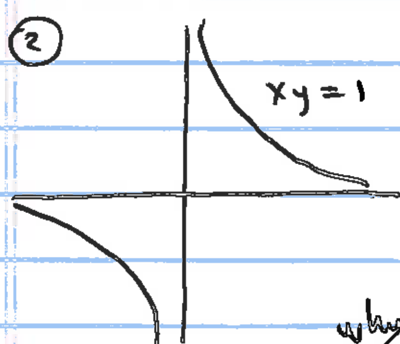
$f(\mathbb{R}) = \{1\}$ not open.

③ $f(x) \equiv 1 : \mathbb{R} \to \mathbb{R}$

$\{1\}$ compact

$f^{-1}(\{1\}) = \mathbb{R}$ not compact.

②




$f: \mathbb{R}^2 \to \mathbb{R}^1$.

$f(x,y) = x$ continuous

$A = \{(x,y) \mid xy = 1\}$ closed in $\mathbb{R}^2$

why? closed

$f(A) = \mathbb{R} - \{0\}$ not closed in $\mathbb{R}^1$

$g(x,y) = xy : \mathbb{R}^2 \to \mathbb{R}^1$ Thm $g$ continuous, $\{1\}$ closed

closed: $A = g^{-1}(\{1\}) \Leftarrow$