March 9, 2018

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CHAP 4

Defn Let (X, d_X), (Y, d_Y) be metric spaces.

$f: E \subseteq X \to Y$, $p \in E'$

We write

$$\lim_{x \to p} f(x) = q \quad \text{or} \quad f(x) \to q \text{ as } x \to p \quad \text{if}$$

$$\exists q \in Y \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \left(\left. \begin{array}{c} 0 < d_X(x,p) < \delta \\ x \in E \end{array} \right\} \Rightarrow d_Y(f(x), q) < \varepsilon \right)$$

(Hidden quantifier $\forall x \in E$)

Thm: $\lim_{x \to p} f(x) = q$

$\Longleftrightarrow$ $\forall \{p_n\}$ in E, $p_n \to p$, $p_n \neq p \ \forall n$, one has $\lim_{n \to \infty} f(p_n) = q$.

Proof:

$\Rightarrow$: Assume $\lim_{x \to p} f(x) = q$.

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \left(\begin{array}{c} 0 < d_X(x,p) < \delta \\ x \in E \end{array} \Rightarrow d_Y(f(x), q) < \varepsilon \right)$$

Let $\{p_n\}$ be sequence in E, $p_n \to p$, $p_n \neq p \ \forall n$

Given δ, $\exists N \in \mathbb{N} \ \forall n \geq N \ d_X(p, p_n) < \delta$

$\forall n \geq N \quad d_X(p, p_n) < \delta \implies d_Y(f(p_n), q) < \varepsilon$

Given $\varepsilon > 0 \ \exists N \ \forall n \geq N \quad d_Y(f(p_n), q) < \varepsilon$

$$\lim_{n \to \infty} f(p_n) = q.$$

($\Leftarrow$ i) Use contrapositive

WTS

$$\text{not}\left(\lim_{x\to p} f(x) = q\right) \Longrightarrow \text{not}\left(\forall \{p_n\} \cdots \lim_{n\to\infty} f(p_n) = q\right)$$

Hidden $\forall x$

$$\text{not}\left(\forall \varepsilon > 0\ \exists \delta > 0\ \left(0 < d_X(x, p) < \delta \Longrightarrow d_Y(f(x), q) < \varepsilon\right)\right)$$
$x \in E$

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x\ 0 < d_X(x, p) < \delta \text{ and } d_Y(f(x), q) \geq \varepsilon.$$
$x \in E$

$\exists \varepsilon > 0$ For each $n \in \mathbb{N}$ $\delta_n = \frac{1}{n}$ $\exists x_n$ $0 < d_X(x_n, p) < \frac{1}{n}$
$x_n \in E$

and $d_Y(f(x_n), q) \geq \varepsilon$.

$\exists$ sequence $\{x_n\}$ $x_n \to p$, $x_n \neq p$ $\lim_{n\to\infty} f(x_n) \neq q$.

Corollary: $\lim_{x\to p} f(x) = q$ and $\lim_{x\to p} f(x) = q'$

$\Longrightarrow q = q'$

p85 4.3, 4.4 HW to read

CONTINUOUS FUNCTIONS

Defn Let $(X, d_X), (Y, d_Y)$ be metric spaces
$f : E \subseteq X \to Y$, $p \in E$

→ f is called continuous at p if

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in E \quad d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$.

→ f is continuous on X if f is continuous at each $p \in X$

Consequently

(1) if $p \in E'$ then
f continuous at $p \iff \lim_{x \to p} f(x) = f(p)$

(2) if $p \notin E'$, then f is automatically continuous at p.

Ex Every function $f : \mathbb{N} \to (X, d)$ is continuous
(standard metric)

Prop Let X, Y, Z be metric spaces

Let $E \xrightarrow{f} F \xrightarrow{g} Z$

$E \subseteq X$, $F \subseteq Y$, $f(E) \subseteq F$ so that

$h(x) = g(f(x)) = (g \circ f)(x)$.

Let f be continuous at $p \in E$, let g be continuous at $f(p)$;
then $h = g \circ f$ is continuous at p.

Proof Let $\varepsilon > 0$ be given

$\exists\, \eta > 0$ s.t.

$$\left.\begin{array}{l} d_Y(q, f(p)) < \eta \\ \forall q \in F \end{array}\right\} \Longrightarrow d_Z(g(q), g(f(p))) < \varepsilon$$

For that given $\eta > 0$ $\exists\, \delta > 0$ s.t.

$$\left.\begin{array}{l} d_X(x, p) < \delta \\ x \in E \end{array}\right\} \Longrightarrow d_Y(f(x), f(p)) < \eta$$

In Summary $\forall \varepsilon > 0$ $\exists\, \delta > 0$ s.t.

$$\begin{array}{l} d(x, p) < \delta \\ x \in E \end{array} \Longrightarrow d_Y(\underbrace{f(x)}_{q}, f(p)) < \eta$$

$$\Longrightarrow d_Z(g(f(x)), g(f(p))) < \varepsilon.$$

Recall $f : X \to Y$

For $A \subseteq X$, $f(A) = \{ f(x) \mid x \in A \} \subseteq Y$

For $B \subseteq Y$, $f^{-1}(B) = \{ x \mid f(x) \in B \} \subseteq X$

Thm: Let $f : (X, d_X) \to (Y, d_Y)$

f is continuous on X

$\Longleftrightarrow$ $\forall\, U$ open Y, $f^{-1}(U)$ is open in X

($\Rightarrow$:) Assume f is continuous on X

Proof $\quad f: X \longrightarrow Y$

$U \subseteq Y \quad U$ open.

WTS: $f^{-1}(U)$ is open.
WTS: every point $p \in f^{-1}(U)$ is an interior pt.

Let $p \in f^{-1}(U)$ be an arbitrary pt

$f(p) \in U$. which is open.

$\exists \varepsilon > 0$ s.t. $N_\varepsilon(f(p)) \subseteq U$.

f cont at p $\Rightarrow \exists \delta > 0 \quad d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$.

$\Rightarrow f(x) \in N_\varepsilon(f(p)) \subseteq U$

$\Rightarrow x \in f^{-1}(U)$

$\forall x \quad x \in N_\delta(p) \Rightarrow x \in f^{-1}(U)$

$N_\delta(p) \subseteq f^{-1}(U)$

p is an interior pt of $f^{-1}(U)$

p is arbitrary $\Rightarrow f^{-1}(U)$ is open.

($\Leftarrow$:) After the break.