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ADDITION & MULTIPLICATION of SERIES

Prop. $\begin{cases} \sum a_n = A, \quad \sum b_n = B \implies \sum (a_n + b_n) = A + B \\ A, B \in \mathbb{R}. \end{cases}$

Ex

$$\left.\begin{matrix}(2 + x + x^2 + 3x^3) \\ \cdot (1 - x + 2x^2 - x^3)\end{matrix}\right\rangle \text{multiply}$$

$$= 2 + x(-2+1) + x^2(4-1+1) + x^3(-2+2-1+3) + \cdots$$

$$\left.\begin{matrix}\sum a_n = a_0 + a_1 + a_2 + a_3 + \cdots \\ \sum b_n = b_0 + b_1 + b_2 + b_3 + \cdot\end{matrix}\right\rangle \text{multiply}$$

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots$$

Called Cauchy product

Define Given $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, define $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

Ex 1 $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = 2$

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \sum_{n'=0}^{\infty} \left(\frac{-1}{2}\right)^{n'} = 1 + \underbrace{\left(\frac{1}{2} - \frac{1}{2}\right)}_{0} + \underbrace{\left(\frac{1}{4} - \frac{1}{4} + \frac{1}{4}\right)}_{1/4} + \underbrace{\left(-\frac{1}{8} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8}\right)}_{0} + \cdots$$

$$= 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(-\frac{1}{2}\right)^{n-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{2^n}\underbrace{\sum_{k=0}^{n}(-1)^{n-k}}_{\begin{cases}1 & n \text{ even } = 2m\\ 0 & n \text{ odd}\end{cases}} = \sum_{m=0}^{\infty}\frac{1}{2^{2m}} = \sum_{m=0}^{\infty}\left(\frac{1}{4}\right)^m = \frac{4}{3}$$

Thm: (Mertens)

If (1) $\sum a_n$ converges absolutely

(2) $\sum a_n = A$

(3) $\sum b_n = B$ $\Big\}$ $A, B \in \mathbb{R}$.

(4)

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

then $\displaystyle\sum_{n=0}^{\infty} c_n = AB$

Proof: $\displaystyle A_n = \sum_{k=0}^{n} a_k \qquad A_n \to A$

$$B_n = \sum_{k=0}^{n} b_k \qquad \beta_n = B_n - B \to 0$$

$$C_n = \sum_{k=0}^{n} c_k \qquad \text{WTS: } C_n \to AB.$$

$$C_n = \sum_{k=0}^{n} c_k$$

$$= a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)$$

$$= a_0B_n + a_1B_{n-1} + \cdots + a_nB_0$$

$$= a_0(\beta_n + B) + a_1(\beta_{n-1}+B) + \cdots + a_n(\beta_0 + B)$$

$$= B\underbrace{(a_0 + a_1 + \cdots + a_n)}_{A_n} + \underbrace{a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0}_{\text{define } \gamma_n}$$

As $n \to \infty$ $\quad A_n \to A \quad$ As $n \to \infty$ Want to show

$\downarrow \qquad\qquad\qquad\qquad \downarrow$

$BA \qquad\qquad\qquad\qquad 0$

$\Longrightarrow C_n \to BA$ provided that $\gamma_n \to 0$.

We know $\sum a_n$ converges absolutely

Let $\sum |a_n| = \alpha \in \mathbb{R}$ ①

$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N \quad |\beta_n| \leq \varepsilon$ ② since $\beta_n \to 0$.

$\forall n \geq N$.

$$|\gamma_n| \leq |\beta_0 a_n + \beta_1 a_{n-1} + \cdots + \beta_N a_{n-N}| + \underbrace{|\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0|}_{\wedge \text{ by ① + ②}}$$

$$|\gamma_n| \leq |\beta_0 a_n + \beta_1 a_{n-1} + \cdots + \beta_N a_{n-N}| + \varepsilon\alpha$$

As $n \to \infty$ $\downarrow$ 0

N is fixed, $\{\beta_n\}$ is fixed

$\beta_0, \beta_1 \cdots \beta_N$ are fixed

as $n \to \infty$ $a_n \to 0$ ($\sum a_n$ convergent)

$$\limsup |\gamma_n| \le \varepsilon\alpha \qquad \forall \varepsilon, \quad \alpha \in \mathbb{R}, \alpha \geq 0$$

$$\limsup |\gamma_n| \le 0$$

$$0 \le \liminf |\gamma_n| \le \limsup |\gamma_n| \le 0$$

$|\gamma_n| \geq 0$

$$\lim |\gamma_n| = 0$$

$$\lim \gamma_n = 0.$$

Ex $\sum a_n = \sum b_n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^p}$ both converge conditionally

Need $0 < p \le \frac{1}{2}$

$$c_n = \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)^p} \cdot \frac{(-1)^{n-k}}{(n-k+1)^p} = (-1)^n \sum_{k=0}^{n} \frac{1}{(k+1)^p(n-k+1)^p}$$

$\not\to 0$

$$|c_n| \geq 2^{2p}\left(\frac{n+1}{n+2}\right)(n+2)^{1-2p}$$

$p \le \frac{1}{2}$

$1-2p \geq 0$

$\lim \frac{n+1}{n+2} = 1 \implies |c_n| \not\to 0$

$\implies \sum c_n$ is divergent

REARRANGEMENTS (Shuffling the terms of a series)

Def Let $\{a_n\}$ be a sequence, $f: \mathbb{N} \to \mathbb{N}$ be a bijection, $k_n = f(n)$.

$a_n' = a_{f(n)} = a_{k_n}$ is called a rearrangement of $\{a_n\}$

Thm: Let $\sum a_n'$ be a rearrangement of $\sum a_n$, and $\sum a_n$ be absolutely convergent.
$\Rightarrow$ $\sum a_n'$ is also convergent, $\sum a_n' = \sum a_n$

Ex $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2$$

Next, we will rearrange the terms of this series so that the new series will converge to $0 \neq \ln 2$.

Obs: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is <u>not</u> absolutely convergent.

$a_n = 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}, \cdots$

Want a rearrangement a_n' s.t. $\sum a_n' = 0$.

$0 < 1$

$0 < 1 - \frac{1}{2}$

$0 < 1 - \frac{1}{2} - \frac{1}{4}$

$0 < 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6}$

$0 > 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} \approx -0.04167$

$0 < 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} \approx 0.29167$

$\vdots$

$0 < 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14}$

$0 > 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16}$

$0 < (\quad '' \quad) + \frac{1}{5} \approx 0.17440$

$0 < (\quad '' \quad) + \frac{1}{5} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22}$

$0 > (\quad '' \quad) + \left(\frac{1}{5} - \frac{1}{18} - \frac{1}{20} - \frac{1}{22} - \frac{1}{24}\right)$

≈ -0.01827

$0 < (\quad \quad + \quad \quad) + \frac{1}{7} \approx 0.12459$

$(\quad '' \quad) + \frac{1}{7} - \frac{1}{26} - \frac{1}{28} - \frac{1}{30} - \frac{1}{32}$

≈ -0.01417.

Rearranging in the following order

1 2 4 6 8 3 10 12 14 16 5 18 20 22 24 7 26 28 30 32 9

Why should this work? PTO

Theorem (Riemann)

Let $\sum a_n$ be a series which converges (conditionally) but **not** absolutely.

Let $-\infty \leq \alpha \leq \beta \leq \infty$ be given

Then $\exists$ a rearrangement $\sum a_n'$ of $\sum a_n$ s.t.

$$s_n' = \sum_{k=0}^{n} a_k'$$

$$\liminf s_n' = \alpha$$
$$\limsup s_n' = \beta.$$

Corollary: (Take $\alpha = \beta$.)
Let $\sum a_n$ be convergent but not absolutely. Let $\alpha \in [-\infty, \infty]$. Then $\exists$ rearrangement $\sum a_n'$ of $\sum a_n$ s.t.

$$\sum_{n=0}^{\infty} a_n' = \alpha.$$

Proof p76-77, is quite similar to the example we have done above with $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, (but done rigorously).

HW to read.