ABSOLUTE CONVERGENCE

Defn A series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ is a convergent series.

Thm: If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof:

Assume $\sum |a_n|$ converges.

CCPS $\forall \varepsilon > 0 \ \exists N, \ \forall m \geq n \geq N \quad \left| \sum_{k=n}^{m} |a_k| \right| < \varepsilon$

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| = \left| \sum_{k=n}^{m} |a_k| \right| < \varepsilon .$$

$\sum a_n$ satisfies CCPS

$\sum a_n$ is convergent.

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but

$\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum \frac{1}{n}$ diverges.

This is called conditional convergence

SUMMATION BY PARTS

Lemma: Given $\{a_n\}, \{b_n\}$

Define $A_n = \sum_{k=0}^{n} a_k$, $A_{-1} = 0$

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Proof:

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (\overbrace{A_n - A_{n-1}}^{a_n}) b_n$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

$$= \sum_{n=p}^{q-1} A_n b_n + A_q b_q - \left(A_{p-1} b_p + \sum_{n=p}^{q-1} A_n b_{n+1} \right)$$

$$= \sum_{n=p}^{q-1} (A_n b_n - A_n b_{n+1}) + A_q b_q - A_{p-1} b_p$$

<u>Thm:</u> Let $\{a_n\}, \{b_n\}$ be given s.t.

(i) $A_n = \sum_{k=0}^{n} a_k$ is a bounded sequence

(ii) $b_0 \geq b_1 \geq b_2 \geq \cdots \; (\geq 0)$ (b_n is a $\downarrow$ sequence)
($\Rightarrow \lim b_n = \inf b_n$)

(iii) $\lim_{n\to\infty} b_n = 0$

Then $\sum_{n=0}^{\infty} a_n b_n$ converges.

<u>Proof:</u> $\{A_n\}$ is bounded, $\exists M$ $|A_n| \leq M$.

Given $\varepsilon > 0$ $\exists N$ s.t. $0 \leq b_N < \frac{\varepsilon}{2M}$

$\forall_{p,q}\, N \leq p \leq q$ (WTS: CCPS for $\sum a_n b_n$)

$$\left| \sum_{k=p}^{q} a_k b_k \right| \overset{\text{Lemma}}{=} \left| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right|$$

$$\leq M \left(\sum_{n=p}^{q-1} |b_n - b_{n+1}| + |b_q| + |b_p| \right)$$

$b_n \geq 0$

$b_n - b_{n+1} \geq 0$

$$= M \left(b_p + b_p - \cancel{b_{p+1}} + \cancel{b_{p+1}} - \cancel{b_{p+2}} + \cdots + \cancel{b_{q-1}} - \cancel{b_q} + \cancel{b_q} \right)$$

$$= 2 b_p M \leq 2 b_N M < \varepsilon$$

CCPS $\sum a_n b_n$ is true

$\sum a_n b_n$ converges.

Corollary Alternating series test (Calculus II)

Let $b_0 \geq b_1 \geq b_2 \geq \cdots b_n \geq b_{n+1} \geq \cdots 0$

$\lim_{n \to \infty} b_n = 0$

Then $\sum_{n=0}^{\infty} (-1)^n b_n$ is convergent

Proof $a_0 = +1$

$a_1 = -1$

$\vdots$

$a_{odd} = -1, \quad a_{even} = +1$

$$\left.\begin{array}{l} A_0 = 1 = A_2 = A_{2k} \\ A_1 = 0 = A_3 = A_{2k+1} \end{array}\right\} \quad \{A_n\} \text{ is a bdd seq.}$$

Then $\Rightarrow \sum a_n b_n = \sum_{n=0}^{\infty} (-1)^n b_n$ convergen.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ convergent

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ convergent $\Longleftrightarrow p > 0$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent $\Longleftrightarrow p > 1$

(Calc II) Exs

$\sum \frac{1}{n} = \infty \qquad \sum \frac{(-1)^{n+1}}{n} < \infty$

$\sum \frac{1}{n^2} < \infty \qquad \sum \frac{(-1)^{n+1}}{n^2} < \infty$

$\sum \frac{1}{\sqrt{n}} = \infty \qquad \sum \frac{(-1)^{n+1}}{\sqrt{n}} < \infty$

Ex Let $m \in \mathbb{N}$, be fixed, $m \geq 2$

We want to study $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(2\pi n/m)}{n^p}$ (Caution $n \neq 0$

<u>Case 1</u> $p > 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

$$|\cos 2\pi n/m| \leq 1$$

By comparison then $\displaystyle\sum_{n=1}^{\infty} \frac{\cos 2\pi n/m}{n^p}$ converges

<u>Case 2</u> $\displaystyle\sum \frac{1}{n^p}$ diverges for $0 < p \leq 1$

<u>Obs</u> $\displaystyle\sum_{k=0}^{m-1} \cos(2\pi k/m) = \sum_{k=1}^{m} \cos(2\pi k/m) = 0.$

<u>Why?</u> $e^{i\theta} = \cos\theta + i\sin\theta$

$1 \neq e^{2\pi i/m} = \sigma$ root of unity

$$\sigma^m = 1 \Rightarrow 0 = \sigma^m - 1 = \underbrace{(\sigma - 1)}_{\neq 0}\underbrace{(\sigma^{m-1} + \sigma^{m-2} + \cdots + \sigma^2 + \sigma + 1)}_{0}$$

$$0 = \text{real}(1 + \sigma + \sigma^2 + \cdots + \sigma^{m-1}) = \sum_{k=0}^{m-1} \cos(2\pi k/m)$$

$$= \sum_{k=1}^{m} \cos(2\pi k/m)$$

((Caution: false for $m=1$; real$(1) = 1 = \cos 2\pi$)

$0 < p \le 1$

To apply the theorem $\quad b_n = \frac{1}{n^p} \downarrow, \geq 0$

$\frac{1}{n^p} \to 0$ when $p > 0$.

$$a_k = \cos\frac{2\pi k}{m} = a_{k+mp} \quad (\forall p \in \mathbb{N})\ (\text{cyclic})$$

$$0 = a_1 + a_2 + \cdots + \underset{1}{\underset{\shortparallel}{a_m}} = \underset{1}{\underset{\shortparallel}{a_0}} + a_1 + \cdots + a_{m-1}$$

$$\left[\ A_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{mp} a_k + \sum_{k=mp+1}^{n} a_k = 0 + \sum_{k=1}^{n_0} a_k .\right.$$

Given n, $\quad n = mp + \underset{\text{remainder}}{\underset{\uparrow}{n_0}}$ for some $p \in \mathbb{N}$.

$0 \le n_0 < m$

$$\sup_{1 \le n < \infty} \{A_n\} = \max\{A_1, A_2, \ldots, A_m\} \le m\ (\text{fixed})$$

$\{A_n\}$ is a bounded sequence.

$n \geq 1 \quad 0 \le b_n = \frac{1}{n^p} \downarrow 0$ for $0 < p \le 1$.

<u>Thm</u> $\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ converges for $0 < p \le 1$.

$$\shortparallel$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p}\left(\cos\frac{2\pi n}{m}\right) \quad \text{for } m \geq 2,\ m \in \mathbb{N}$$

Caution false for $m = 1$: $\quad \sum_{n=1}^{\infty} \frac{\cos(2\pi n/1)}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p} = +\infty$

$0 < p \le 1$