CONTINUE Chap I

1.10 An ordered set S is said to have LUB-property if
Every set $E \subseteq S$, $E \neq \emptyset$, E bounded above, one has $\exists \sup E \in S$.

1.11 **Thm:** Let S be an ordered set with the LUB property. Let $B \subseteq S$, $B \neq \emptyset$, B be bounded below, then $\inf B$ exists in S, and $\inf B = \sup L$, where $L = \{\beta \mid \beta \text{ is a lower bd for } B\}$

i.e. LUB-property $\Longrightarrow$ GLB property.

Remark: S is a set, not necessarily $\mathbb{R}$, or $\mathbb{Q}$.

1.11 Proof $B \subseteq S$, $B \neq \emptyset$, B bounded below (Given)

Let $L = \{\beta \mid \beta \text{ is a lower bound of } B\}$

$L \neq \emptyset$ (B is bd below)

L is bounded above since $\exists x_0 \in B \neq \emptyset$.

$\forall \beta \in L \quad \beta \leq x_0$

x_0 is an upper bd for L

LUB-property $\Longrightarrow \exists \alpha = \sup L$.

Claim: $\alpha = \sup L = \inf B$.

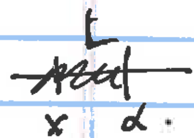
To show (i) α is a lower bd for B.

To show (ii) α is the greatest lower bound for B.

(i) Suppose not$(\forall x \in B,\ x \geq \alpha)$

$\exists x \in B \quad x < \alpha = \sup L.$

x can't be an upper bound for L.



$\exists\, l \in L$ s.t. $l > x$

Contradiction. $\left\{ \begin{array}{l} \therefore\ l \in L \quad l \text{ is a lower bound for } B. \\ l \text{ is not a lower bound for } B \text{ since } l > x, \ x \in B. \end{array} \right.$

Hence α is a lower bd for B.

(ii) Take any $\gamma > \alpha = \sup L$.

$\gamma \notin L =$ the set of all lower bds of B.

γ is not a lower bound for B.

$\implies \alpha = \inf B.$

Fields Read p 5 – 8

1.19 Thm: There exists an ordered field $\mathbb{R}$ which has the LUB property and $\mathbb{Q} \subseteq \mathbb{R}$.

Several constructions for $\mathbb{R}$

- Dedekind cuts
- Cauchy sequences (Cantor)

<u>Thm:</u> (i) $\forall x, y \in \mathbb{R}$, $x > 0$, $\exists n \in \mathbb{N}$ $nx > y$

(ii) $\forall x, y \in \mathbb{R}$, $x < y$, then $\exists p \in \mathbb{Q}$ s.t. $x < p < y$.

(i) Archimedean property
(ii) Density of rationals in real numbers.

<u>Proof:</u> (i) is the same as in 3770 class
Read p 9 of Rudin. (Clear enough)

our proof of (ii) is slightly different from Rudin.
(We will integrate Well-orderedness Axiom into this proof, as a consequence of LUB-prop.)

Given $x, y \in \mathbb{R}$, $x < y$.

$y - x > 0$

$\exists n \in \mathbb{N}$, s.t. $n(y-x) > 1$ (Arch. Prop.)

Let $A = \{ z \in \mathbb{Z} \mid z \leq nx \}$ ← fixed #

$A \neq \emptyset$ since $\exists m_1 \in \mathbb{N}$ s.t. $m_1 > -nx$ (Arch.P.)

$-m_1 < nx$

$-m_1 \in A$.

A is bounded above by nx.

LUB-Prop: $\exists$ $\alpha = \sup A$ in $\mathbb{R}$.

$\alpha = \sup A$

$\alpha - 1$ is not an upper bd for A.

$\exists\, k \in A\ (k \in \mathbb{Z})$ s.t. $\alpha - 1 < k \leq \alpha$.

$\alpha \leq nx$, $k \leq nx$ (since α is least u. bound, nx is an u. bound.)

$\sup A = \alpha < k+1$

$k+1 \notin A$

$nx < k+1$ (since all elements of A are $\leq nx$, & vice versa.)

$nx < k+1 \leq 1 + nx < n(y-x) + nx = ny$

since $n(y-x) > 1$

$nx < k+1 < ny$

Choose $k+1 = m$.

$k \in \mathbb{Z} \Rightarrow m \in \mathbb{Z}$.

$n \in \mathbb{N}$ as chosen earlier

$nx < m < ny$

$x < \frac{m}{n} < y$

$p = m/n \in \mathbb{Q}$. #