Ex

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

$$a_n = \begin{cases} \left(\frac{1}{2}\right)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ \frac{1}{(n/2)!} & \text{if } n \text{ is even.} \end{cases}$$

$n=0$

$$1,\ 1,\ 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{24} + \frac{1}{16} + \cdots$$

Root test $\sqrt[n]{\left(\frac{1}{2}\right)^{\frac{n-1}{2}}} = \sqrt[n]{|a_n|}$ when $n$ is odd

$$= \left(\frac{1}{2}\right)^{\frac{n-1}{2n}} = \left(\frac{1}{\sqrt{2}}\right)^{\frac{n-1}{n}} = \left(\frac{1}{\sqrt{2}}\right)^{1-\frac{1}{n}} \longrightarrow \frac{1}{\sqrt{2}}$$

when $n$ is even $n = 2m$ or $m = n/2$

$$\sqrt[n]{\frac{1}{(n/2)!}} = \left((n/2)!\right)^{-\frac{1}{n}} = \left(\left(\left(\frac{n}{2}\right)!\right)^{\frac{2}{n}}\right)^{-\frac{1}{2}} = \left((m!)^{\frac{1}{m}}\right)^{-\frac{1}{2}}$$

$$\sim \left(\frac{m}{e}\sqrt[2m]{2\pi m}\right)^{-\frac{1}{2}} \longrightarrow 0 \text{ since } m \to \infty.$$

$$\boxed{\text{Stirling's Formula} \quad n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad (n!)^{\frac{1}{n}} \sim \frac{n}{e}\sqrt[2n]{2\pi n}.}$$

$$\liminf \sqrt[n]{|a_n|} = 0$$

$$\limsup \sqrt[n]{|a_n|} = \frac{1}{\sqrt{2}} < 1$$

$\sum a_n$ converges by root test.

$$\frac{a_{2n+1}}{a_{2n}} = \frac{(1/2)^n}{1/n!} = \frac{n!}{2^n} \longrightarrow +\infty$$

$$= \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{2^n}$$

$$\frac{a_{2n}}{a_{2n-1}} = \frac{1/(n!)}{(1/2)^{n-1}} = \frac{2^{n-1}}{n!} \longrightarrow 0$$

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| = +\infty \quad \text{Ratio test fails}$$

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| = 0$$

$$\left.\begin{aligned} \liminf \sqrt[n]{|a_n|} &= 0 \\ \limsup \sqrt[n]{|a_n|} &= \frac{1}{\sqrt{2}} \end{aligned}\right\} \text{Root test} \Rightarrow \text{convergent}$$

Read example 3.35 p 67

**Thm** Let $\{c_n\}$ be a sequence of positive reals

$$\liminf \frac{c_{n+1}}{c_n} \le \underbrace{\liminf \sqrt[n]{c_n} \le \limsup \sqrt[n]{c_n}}_{\text{obvious}} \le \limsup \frac{c_{n+1}}{c_n}. \quad 1$$

**Proof:** proof of 1

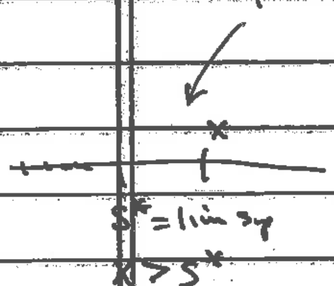
WTS $\limsup \sqrt[n]{c_n} \le \limsup \frac{c_{n+1}}{c_n}$.

**Case 1** $\limsup \frac{c_{n+1}}{c_n} = +\infty$, nothing to prove.

**Case 2** $\limsup \frac{c_{n+1}}{c_n} = \alpha \in \mathbb{R}$.

Let $\alpha < \beta < \infty$.

prop 3.17 $\exists N \ \forall n \ge N$ $\quad \frac{c_{n+1}}{c_n} < \beta$




$$c_{n+1} < \beta c_n.$$
$$c_{n+2} < \beta c_{n+1} < \beta^2 c_n.$$
$$\vdots$$
$$c_{n+k} < \beta^k c_n.$$

$m = N + k$
$$c_m < \beta^{m-N} c_N$$

$$\sqrt[m]{c_m} < \left(\frac{\beta^m}{\beta^N} c_N\right)^{\frac{1}{m}} = \beta \cdot \sqrt[m]{\beta^{-N}} \ \sqrt[m]{c_N} \longrightarrow \beta$$

N fixed

$m \to \infty$

$$\limsup \sqrt[m]{c_m} \le \beta \ (\forall \beta > \alpha) \implies \limsup \sqrt[m]{c_m} \le \alpha.$$

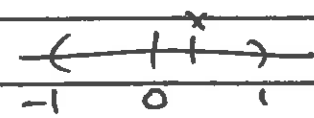
#

## Power Series

$$\sum_{n=0}^{\infty} c_n (x-a)^n \text{ in } \mathbb{R} \; ; \quad \sum_{n=0}^{\infty} d_n (z-z_0)^n, \text{ in } \mathbb{C}$$

Convergency depends on the choice of $x$ (resp. $z$)

(Ex) $\sum_{n=0}^{\infty} x^n$ converges if $|x|<1$
diverges if $|x| \geq 1$




$1 = R =$ radius of convergence

**Thm:** Given $\sum_{n=0}^{\infty} c_n (x-a)^n$, $\alpha = \limsup_{n \to \infty} \sqrt[n]{|c_n|}$

$R = \frac{1}{\alpha}$. $\left( \begin{array}{l} \alpha = 0 \Rightarrow R = +\infty \\ \alpha = \infty \Rightarrow R = 0 \end{array} \right.$

Proof is an immediate consequence of Root test

$\sum c_n (x-a)^n$ converges for $|x-a| < R$.
" diverges for $|x-a| > R$
$\sum c_n (x-a)^n$ inconclusive for $|x-a| = R$.

(Ex) ① $\sum \frac{(x-a)^n}{n!} = e^{x-a}$ $\quad R = \infty$

$\sum \frac{x^n}{n}$ $\quad R = 1$

$\alpha = \limsup_{n \to \infty} \sqrt[n]{\frac{1}{n}} = 1$

$|x| < 1$ converges ; $x = 1$ diverges
$|x| > 1$ diverges ; $x = -1$ converges

$p \in \mathbb{R}$

2) $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$ $\sqrt[n]{\frac{1}{n^p}} = \left(\frac{1}{\sqrt[n]{n}}\right)^p \longrightarrow 1$

$R = 1/1 = 1$

$\sum_{n=1}^{\infty} x^n/n^p$ converges when $|x| < 1$; diverges for $|x| > 1$.

$x = 1$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$ Convergent $\iff$ $p > 1$

$x = -1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ convergent $\iff$ $p > 0$