Thm: If $s_n \le t_n \ \forall n \geq N$ for some N fixed, then

$$\liminf s_n \le \liminf t_n$$
$$\limsup s_n \le \limsup t_n. \qquad \text{(No proof)}$$

Number e

Defn $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. Why does it converge?

$$s_n = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots + \frac{1}{n!} < \underbrace{1 + 1 + \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^{n-1}}}_{< 1}}_{< 3}$$

$0 \le s_n < 3$

$s_{n+1} = s_n + \frac{1}{(n+1)!}$ $\qquad s_{n+1} \geq s_n \ \forall n.$

Bounded + monotone $\Rightarrow$ convergent.

Prop $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

<u>Proof:</u> $s_n = \sum_{k=0}^{n} \frac{1}{k!}$ $\qquad t_n = \left(1+\frac{1}{n}\right)^n$

① $t_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \left(\frac{1}{n}\right)^k 1^{n-k} \binom{n}{k}$

$= 1 + \binom{n}{1}\left(\frac{1}{n}\right)^1 + \binom{n}{2}\left(\frac{1}{n}\right)^2 + \dots + \binom{n}{n}\left(\frac{1}{n}\right)^n$

$= 1 + 1 + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3} + \dots$

$+ \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k + \dots + \frac{1}{n!}\frac{n(n-1)\cdots 1}{n^n}$

$t_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots$

$+ \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) + \dots$

$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

$t_n \le 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} = s_n.$ $\quad \left(\left(1-\frac{1}{n}\right) < 1 \text{ etc.}\right)$

$\forall n \quad t_n \le s_n.$

$\limsup t_n \le \limsup s_n = \lim s_n = e.$

② If $n \ge m$

$t_n \ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \dots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)$

truncated at $(m+1)^{\text{st}}$ term.

fixed
$m \le n \to \infty$.

$\underbrace{\liminf t_n}_{\text{fixed \#}} \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = S_m$

$\liminf t_n \geq \liminf s_n = e$.

$e \le \liminf t_n \le \limsup t_n \le e$

$\Rightarrow \lim t_n$ exists $= e$.

Obs: $\quad 0 < e - s_n < \frac{1}{n(n!)}$

Why?

$$s_n = \sum_{k=0}^{n} \frac{1}{k!}$$

$$e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$$

$$= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right.$$

$$< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\right)$$

$$= \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n!} \frac{1}{(n+1) - 1} = \frac{1}{n(n!)}$$

Consequence ① $\sum \frac{1}{k!} \to e$ really fast.

Thm: e is irrational

Prof suppose not, i.e., $e = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$.
$q \neq 0$.
$q > 0$

$$0 < (e - s_q) < \frac{1}{q(q!)}$$

$$0 < q!(e - s_q) < \frac{1}{q}$$

$$q!\, e = q! \frac{p}{q} \in \mathbb{Z}.$$

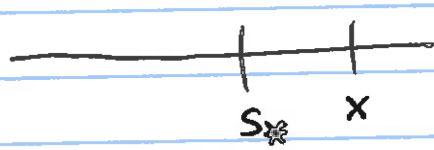
$$q!\, s_q = q!\left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{q!}\right) \in \mathbb{Z}.$$

$$q!(e - s_q) \in \mathbb{Z}.$$

Contradiction.

Thm 3.17 b.

$s^* = \limsup s_n$, if $x > s^*$ then
$\exists N \in \mathbb{N}$ $\forall n \geq N$ $s_n < x$



i.e. one can't have infinitely many n s.t. $s_n \geq x$.

Prof Hw to read.

Thm Root Test

Let $\sum a_n$ be given, $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

(a) If $\alpha < 1$ then $\sum a_n$ converges
(b) if $\alpha > 1$ then $\sum a_n$ diverges
(c) If $\alpha = 1$ then inconclusive

Proof: (a) $\alpha < 1 \implies \exists \beta \quad \alpha < \beta < 1$

By Thm 3.17b $\exists N \; \forall n \geq N \quad \sqrt[n]{|a_n|} < \beta$.

$$|a_n| < \beta^n$$

$$\sum_{n=1}^{\infty} \beta^n \text{ converges since } 0 < \beta < 1$$

Comparison test $\implies \sum a_n$ converges.

(b) If $\alpha > 1 \; \exists$ subsequence $\sqrt[n_k]{|a_{n_k}|} \to \alpha > 1$ contradic

Recall

Thm: $\sum a_n$ convergent $\implies \lim a_n = 0$.

(c) $\sum \frac{1}{n} = +\infty \qquad \sqrt[n]{\frac{1}{n}} \to 1 = \alpha$

$\sum \frac{1}{n^2} < +\infty \qquad \sqrt[n]{\frac{1}{n^2}} = \left(\frac{1}{\sqrt[n]{n}}\right)^2 \to 1 = \alpha$.

Thm (Ratio Test) Let $\sum a_n$ be given $a_n \neq 0$ $\forall n$.

Let $\alpha = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

(a) if $\alpha < 1$ then $\sum a_n$ converges

(b) If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ $\forall n \geq N$ for some fixed N then $\sum a_n$ diverges

Proof (a) Since $0 \leq \alpha < 1$ $\exists \beta$ $\alpha < \beta < 1$.

$\exists N$ $\forall n \geq N$ $\left|\frac{a_{n+1}}{a_n}\right| < \beta$. Thm 3.17

$$|a_{n+1}| < \beta |a_n|$$
$$|a_{n+2}| < \beta^2 |a_n|$$
$$\vdots$$
$$|a_{n+k}| < \beta^k |a_n| \quad k > 0$$

Let $n = N$ $\forall k > 0$ $|a_{N+k}| < \beta^k |a_N|$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$$

a finite sum

compare to $\sum_{k=0}^{\infty} \beta^k |a_N|$

geometric series

fixed value

$\Rightarrow \sum a_n$ is convergent