Some special sequences

Prop:

$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad \text{if } p>0$$

$$\lim_{n\to\infty} \sqrt[n]{p} = 1 \quad \text{if } p>0$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0 \quad \alpha\in\mathbb{R},\ p>0$$

$$\lim_{n\to\infty} |x|^n = 0 \quad \text{if } |x|<1$$
$$= 1 \quad \text{if } |x|=1$$
$$= +\infty \quad \text{if } |x|>1.$$

$$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } -1<x<1 \\ 1 & \text{if } x=1 \\ \text{diverges} & \text{if } x\le -1 \\ \infty & \text{if } x>1 \end{cases}$$

In $\mathbb{R}$ or $\mathbb{C}$ **SERIES** Given $\{a_n\}_{n=1}^{\infty}$

$$\sum_{n=p}^{q} a_n = a_p + a_{p+1} + \cdots + a_q \quad \text{if } p\le q$$

Given $\sum_{n=1}^{\infty} a_n$, how do we discuss its convergence?

Remark: Need $\mathbb{R}$, $\mathbb{C}$ or a vector space in order to add.

Def Partial Sums $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$

Def We say $\sum_{n=1}^{\infty} a_n$ converges & equals to s if

the partial sums $s_n \to s$.

$\sum_{n=1}^{\infty} a_n$ is divergent if $\{s_n\}$ is a divergent sequence.

Prop: (CCPS) (Cauchy criterion for partial sums)

$\sum_{n=1}^{\infty} a_n$ is convergent $\iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n, m \geq N$, $m \geq n$

$\left| \sum_{k=n}^{m} a_k \right| < \varepsilon$.

Proof:

$\sum_{n=1}^{\infty} a_n$ is convergent

$\iff$ $\{s_n\}$ is convergent where $s_n = \sum_{k=1}^{n} a_k$.

$\iff$ $\{s_n\}$ is Cauchy

$\iff$ $\forall \varepsilon > 0 \ \exists N \ \forall n, m \quad m \geq n \geq N \quad |s_m - s_{n-1}| < \varepsilon$

$\iff$ $\forall \varepsilon > 0 \ \exists N \ \forall n, m \quad m \geq n \geq N$

$$\varepsilon > |s_m - s_{n-1}| = \left| \sum_{k=1}^{m} a_k - \sum_{k=1}^{n-1} a_k \right| = \left| \sum_{k=n}^{m} a_k \right|$$

<u>Corollary</u> $\sum a_n$ is convergent $\Longrightarrow \lim a_n = 0$

$$\forall \varepsilon > 0 \ \exists N \ \forall n = m \geq N \quad \left|\sum_{k=n}^{m} a_k\right| = |a_n| < \varepsilon .$$

Converse is false:
$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

<u>Thm 3.25</u> Comparison test

(a) If $|a_n| \leq c_n, \ \forall n \geq N_0$ then

$$\sum c_n \text{ converges} \Longrightarrow \sum a_n \text{ converges}$$
$$\sum c_n \text{ converges} \Longrightarrow \sum |a_n| \quad \text{"}$$

(b) If $a_n \geq d_n \geq 0 \ \forall n \geq N_0$ then

$$\sum d_n \text{ diverges} \Longrightarrow \sum a_n \text{ diverges}$$

<u>Proof</u> $\sum c_n$ converges $\Longrightarrow \{c_n\}$ satisfies CCPS

$$\Longrightarrow \forall \varepsilon > 0 \ \exists N \ \forall n, m \quad N \leq n \leq m \quad \left|\sum_{k=n}^{m} c_k\right| < \varepsilon$$

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k = \left|\sum_{k=n}^{m} c_k\right| < \varepsilon .$$
$$c_k \geq 0$$

$\{a_n\}$ satisfies CCPS
$\{a_n\}$ converges.

Series of non-negative terms

Obs $\forall n$ $a_n \geq 0 \implies S_n = \sum_{k=1}^{n} a_k$ is increasing.

$$S_1 \leq S_2 \leq S_3 \leq \dots \leq S_k \leq S_{k+1} \leq \dots$$

Thm: 3.14 p55 Recall An increasing sequence in $\mathbb{R}$ is convergent if and only if it is bounded.

$\forall n$ $S_{n+1} \geq S_n \Rightarrow \left(\sup\{S_n \mid n \in \mathbb{N}\} = s \iff \lim S_n = s. \right)$

Prop: If $0 \lneq x < 1$ then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Proof

$$S_n = 1 + x + x^2 + \dots + x^n = \frac{1-x^{n+1}}{1-x} \xrightarrow[n\to\infty]{} \frac{1}{1-x}.$$

$|x|^n \to 0$

3.27 Thm

Let $a_1 \geq a_2 \geq a_3 \geq a_4 \cdots \geq a_k \geq a_{k+1} \geq \cdots 0$

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges.}$$

$$a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots$$

Ex 1

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \iff p > 1$$

Use Thm 3.27 $\quad a_n = \frac{1}{n^p}$

$$2^k a_{2^k} = 2^k \cdot \frac{1}{(2^k)^p} = \frac{2^k}{2^{kp}} = \frac{1}{2^{kp-k}} = \frac{1}{(2^{p-1})^k}$$

Geometric series $\quad \displaystyle\sum_{k=0}^{\infty} \frac{1}{(2^{p-1})^k}$ converges $\iff 2^{p-1} > 1$.

$$\iff p-1 > 0$$

$$\iff p > 1.$$

Ex

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} \iff p > 1$$

$$2^k a_{2^k} = \cancel{2^k} \cdot \frac{1}{\cancel{2^k}(\ln 2^k)^p} = \frac{1}{(k \ln 2)^p} = \frac{1}{(\ln 2)^p} \cdot \frac{1}{k^p}$$

When does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge? $\iff p > 1$

$$\sum \frac{1}{n(\ln n)(\ln\ln n)} = \infty$$

$$\sum \frac{1}{n \ln n (\ln \ln n)^2} < \infty$$

HW to check w/ Thm 3.27 & other examples we did.

<u>Proof of Thm 3.27</u>

Hypo. $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq a_{k+1} \geq \cdots \geq 0$

<u>Want</u> $\sum_{n=1}^{\infty} a_n$ converges $\iff$ $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n.$$

$$t_k = a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}.$$

(i) <u>WTS</u> · for $n < 2^k$, $S_n \leq t_k$

$$S_n \leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_8 + \cdots + a_{15}) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1})$$

$a_3 \leq a_2$ $\quad a_4 \geq a_5 \geq a_6 \geq a_7$

$$\leq a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots + 2^k a_{2^k}.$$

$$= t_k.$$

* Recall : An increasing sequence converges iff it is bdd.

If $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges $\implies$ t_k is a bdd $\nearrow$ seq.

$\implies$ S_n is a " $\nearrow$ seq.

$\implies$ $\sum a_n$ converges.

(ii) For $n > 2^k$ WTS $2s_n \geq t_k$.

we discarded some terms.

$$s_n \geq a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k})$$

$a_3 \geq a_4$

$$\geq \frac{1}{2} a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} = \frac{1}{2} t_k.$$

So $s_n \geq \frac{1}{2} t_k$. when $n > 2^k$

$\sum a_n$ converges $\Longrightarrow$ s_n is a bdd $\uparrow$ sequence

$\Longrightarrow$ t_k is " " " "

$\Longrightarrow$ $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.