<u>Defn</u> A metric space (X, d) is called <u>complete</u> if every Cauchy sequence in (X, d) converges.

<u>IMPORTANT THM:</u>

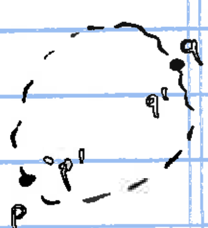
(1) Every compact metric space is complete.
(2) $\mathbb{R}^k$ with the standard metric is complete.

<u>Still we need to establish</u>:

<u>Prop</u>: (a) if $\bar{E}$ is the closure of E, then
$$\operatorname{diam} E = \operatorname{diam} \bar{E}$$
(b) If the K_n is compact $\phi \neq K_{n+1} \subseteq K_n$ and
$$\lim_{n\to\infty} \operatorname{diam} K_n = 0,$$
then $\bigcap_{n\geq 1}^{\infty} K_n = \{p\}$ for some unique $p \in X$

Proof
(a) $E \subseteq \bar{E}$
$\operatorname{diam} E \leq \operatorname{diam} \bar{E}$ $\quad \left(\operatorname{diam} A = \sup \{ d(x,y) \mid x, y \in A \} \right)$
if $A \subseteq B$
$\operatorname{diam} A \leq \operatorname{diam} B$

Let $\varepsilon > 0$ be given.
Choose any $p, q \in \bar{E}$
$\exists \; p', q' \in E$ $\quad d(p', p) < \varepsilon$
$\quad d(q', q) < \varepsilon$



$$d(p,q) \le d(p,p') + d(p',q') + d(q',q)$$
$$< 2\varepsilon + d(p',q')$$

$$\le 2\varepsilon + \operatorname{diam} E$$

$\forall\, p,q \in \bar{E}$ $\quad d(p,q) \le \underbrace{2\varepsilon + \operatorname{diam} E}_{\text{fixed \#}}$

$\downarrow$ sup.

$$\operatorname{diam}(\bar{E}) \le 2\varepsilon + \operatorname{diam} E \quad \text{for any fixed } \varepsilon > 0$$

$$\operatorname{diam}(\bar{E}) \le \operatorname{diam}(E)$$
$$\operatorname{diam}(E) \le \operatorname{diam}(\bar{E}) \quad \text{done earlier}$$
$$\Rightarrow \operatorname{diam} E = \operatorname{diam} \bar{E}.$$

(b) $\emptyset \ne K_{n+1} \subseteq K_n$

define $K = \bigcap_{n=1}^{\infty} K_n \ne \emptyset$ (prop 2.36 & its corollary)

Suppose $\exists\, p,q \in K$, $p \ne q$

$K \subseteq K_n \ \forall n.$

$0 < d(p,q) \le \operatorname{diam} K \le \operatorname{diam} K_n \ \forall n.$

$\downarrow$ as $n \to \infty$

0

Contradiction.

$$\bigcap_{n=1}^{\infty} K_n = \{p\}$$

HW to read 3.13, 3.14 & its proof p 55.
monotone sequences.

Recall we defined convergence $p_n \to p \in X$

Def Let $\{s_n\}$ be a sequence in $\mathbb{R}$.

- If $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N \quad x_n \geq M$,
  then we say $s_n \to +\infty$
  "s_n diverges to ∞".
- If $\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N \quad x_n \leq M$
  then we say $s_n \to -\infty$
  "s_n diverges to $-\infty$".

Defn Let (s_n) be a sequence in $\mathbb{R}$.

$$E = \{ x \in \mathbb{R} \cup \{+\infty, -\infty\} \mid \exists \text{ subsequence } (s_{n_k}) \text{ of } (s_n) \text{ s.t. } s_{n_k} \to x \}$$

$$\sup E = \limsup_{n\to\infty} s_n = \overline{\lim_{n\to\infty}}\, s_n = s^*$$

$$\inf E = \liminf_{n\to\infty} s_n = \underline{\lim_{n\to\infty}}\, s_n = s_*$$

Ex ① If $s_n \to s \in \mathbb{R}$: $s = \lim_n s_n = \limsup_n s_n = \liminf_n s_n$

② t_n: $1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, \ldots$

$$E = \{0, \infty\}$$

$\limsup t_n = +\infty$

$\liminf t_n = 0$

③ $\sin \frac{\pi n}{4}$ $\quad \limsup = 1 \quad \liminf = -1.$

④ $(-1)^n \left(\frac{n^2}{n+1}\right)$ $\quad \limsup = +\infty \quad \liminf = -\infty$

⑤ $(-1)^n \left(\frac{n}{n^2+1}\right)$ $\quad \limsup = 0 = \liminf$

Ex 6 List all positive rational numbers.

$$\begin{array}{cccccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \cdots
\end{array}$$

Diagonally take all p/q <u>WITHOUT</u> discarding any.

(S_n) $\quad \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \frac{2}{4}, \frac{3}{3}, \frac{4}{2}, \frac{5}{1}, \frac{1}{6} \cdots$

Every positive rational number p/q appears in this sequence infinitely many times $\quad p/q = mp/mq$

$m, p, q \in \mathbb{N}$.

Claim The set of all subsequential limits of (S_n) is $E = [0, \infty]$ in $\mathbb{R} \cup \{\infty\}$

① $\underbrace{\liminf S_n = 0}_{1/n \to 0}$, $\quad \limsup S_n = +\infty$

$n/1 \to +\infty$.

② Let $x \in \mathbb{R}$, $x > 0$.

$\mathbb{Q}$ is dense in $\mathbb{R}$, Thm 1.20 p.9

$\forall n \in \mathbb{N} \ \exists p_n \in \mathbb{Q}, p_n > 0$ s.t. $\quad x - \frac{1}{n} < p_n < x + \frac{1}{n}$

Hence $\quad p_1, p_2, p_3, \cdots p_n \longrightarrow x$, $\quad p_n$ is a positive rational number

WTS: (p_k) is a subsequence (S_{n_k}) of (S_n).

Choose $p_1 = S_{n_1}$ since S_n contains all of positive rationals.

Assume $p_1, p_2, \cdots p_k$ are chosen $p_k = S_{n_k}$, $n_1 < n_2 < \cdots < n_k$.

Since p_{k+1} appears in the sequence (S_n) infinitely many times $\exists n_{k+1}$ s.t. $p_{k+1} = S_{n_{k+1}}$ and $n_{k+1} > n_k$.