Feb 19, 2018

①

Ex $p_n = \sin \frac{\pi n}{2}$: $0, 1, 0, -1, 0, 1, 0, -1 \ldots$

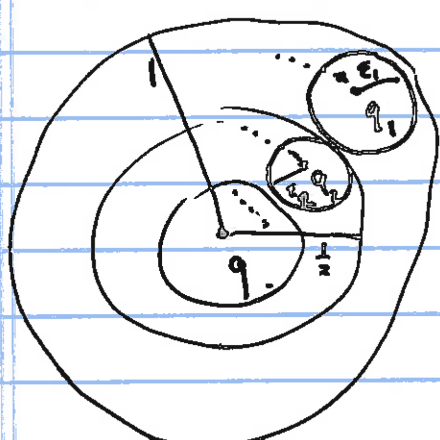
$L = \{0, 1, -1\}$

3.7 **Thm:** Given a sequence $\{p_n\}$ in (X, d), the set

$$L = \{ p \in X \mid \exists \text{ subsequence } \{p_{n_k}\} \text{ of } \{p_n\} \text{ s.t. } \lim_{k \to \infty} p_{n_k} = p \}$$

is a closed set.

Proof: Let $q \in L'$, want to show $q \in L$.



$\forall k \in \mathbb{N}\ \exists q_k \in L$ s.t.

$$q_k \in N_{\frac{1}{k}}(q) \cap (L - \{q\}) \neq \emptyset$$

$$\exists \varepsilon_k > 0 \quad N_{\varepsilon_k}(q_k) \subseteq N_{\frac{1}{k}}(q)$$

$q_1 \in L$, there exists a subsequence $\{p_{n_\ell}\}$ of $\{p_n\}$, converging to q_1

For infinitely many n_ℓ $\quad p_{n_\ell} \in N_{\varepsilon_1}(q_1) \subseteq N_1(q)$

choose any of them and call it p_{m_1}

$q_2 \in L$, there exists a subsequence $\{p_{n_\ell}\}$ of $\{p_n\}$ converging to q_2.

For infinitely many n_ℓ $\quad p_{n_\ell} \in N_{\varepsilon_2}(q_2) \subseteq N_{\frac{1}{2}}(q)$

$\exists\, m_2 > m_1$ s.t. $\quad p_{m_2} \in N_{\varepsilon_2}(q_2) \subseteq N_{\frac{1}{2}}(q)$

Inductively we have $m_1 < m_2 < \dots < m_\ell$,

and $\quad p_{m_i} \in N_{\varepsilon_i}(q_i) \subseteq N_{\frac{1}{i}}(q)$. $\quad 1 \le i \le \ell$

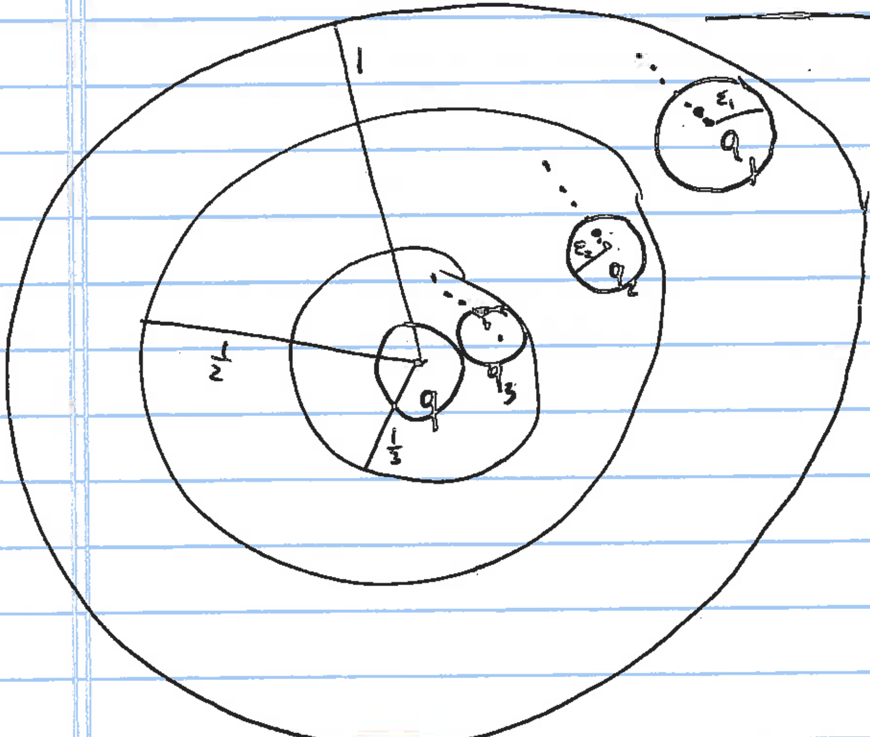
Choose $m_{\ell+1} > m_\ell$ s.t.

$$p_{m_{\ell+1}} \in N_{\varepsilon_{\ell+1}}(q_{\ell+1}) \subseteq N_{\frac{1}{\ell+1}}(q). \quad \forall \ell.$$

$\forall \ell \in \mathbb{N}$, $d(p_{m_\ell}, q) < \frac{1}{\ell} \implies p_{m_\ell} \to q$

$q \in L$. #

main idea



Start with: $\underset{\in L}{q_k} \to q \in L'$

To obtain $p_{m_\ell} \to q' \in L$

Cauchy Sequences

Defn Let $\{p_n\}$ be a sequence in (X, d).

$\{p_n\}$ is called Cauchy if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}, \ \forall n, m \geq N \quad d(p_n, p_m) < \varepsilon$$

Defn $E \subseteq (X, d)$

$$\operatorname{diam}(E) = \sup \{ d(p,q) \mid p, q \in E \}$$

Prop: (a) if $\overline{E}$ is the closure of E, then
$$\operatorname{diam} E = \operatorname{diam} \overline{E}$$

(b) If $\forall n \in \mathbb{N}$ K_n is compact, $K_{n+1} \subseteq K_n$, and

$$\lim_{n \to \infty} \operatorname{diam}(K_n) = 0, \text{ then}$$

$$\bigcap_{n=1}^{\infty} K_n = \{p\} \text{ is a single pt.}$$

(Proof Later
on 2/21/18 Wednesday

VERY IMPORTANT

(3.11) Thm

(a) In every metric space (X, d)
$\{p_n\}$ is convergent $\Longrightarrow$ $\{p_n\}$ is Cauchy

(b) In every compact metric space (X, d)
$\{p_n\}$ is Cauchy $\Longrightarrow$ $\{p_n\}$ is convergent to a limit $p \in X$

(c) In $\mathbb{R}^k$, every Cauchy sequence converges

(a) $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \quad d(p_n, p) < \frac{\varepsilon}{2}$,

since $p_n \longrightarrow p$.

$\forall m, n \geq N \quad d(p_m, p_n) \leq d(p_m, p) + d(p, p_n)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. #

PTO

(b) Let $\{p_n\}$ be a Cauchy sequence.

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}, \ \forall n, m \geq N \quad d(p_n, p_m) < \varepsilon.$

$E_N = \{p_n \mid n \geq N\}, \quad \operatorname{diam}(E_N) \leq \varepsilon$

$\varepsilon \geq \operatorname{diam} \overline{E_N} = \operatorname{diam} E_N$ (previous prop).

$\lim_{N\to\infty} \operatorname{diam} \overline{E_N} = 0$ (*)

$\mathbb{X}$ compact, $\overline{E_N}$ closed $\Rightarrow \overline{E_N}$ compact

$E_{N+1} \subseteq E_N$

$\overline{E_{N+1}} \subseteq \overline{E_N}$

$\bigcap_{N=1}^{\infty} \overline{E_N} = \{p\}$ a single pt. (previous proposition)

$p \in \overline{E_N} \quad \forall N$

Let $\varepsilon > 0 \ \exists N_0 \in \mathbb{N} \quad \forall N \geq N_0$

$\operatorname{diam} \overline{E_N} = \operatorname{diam} E_N < \varepsilon$ from (*)

$p \in \overline{E_N}$ already

$\forall n \geq N \quad p_n \in E_N \subseteq \overline{E_N}$

$d(p, p_n) \leq \operatorname{diam} \overline{E_N} < \varepsilon \quad \forall n \geq N_0.$

$\lim_{n\to\infty} p_n = p.$ #

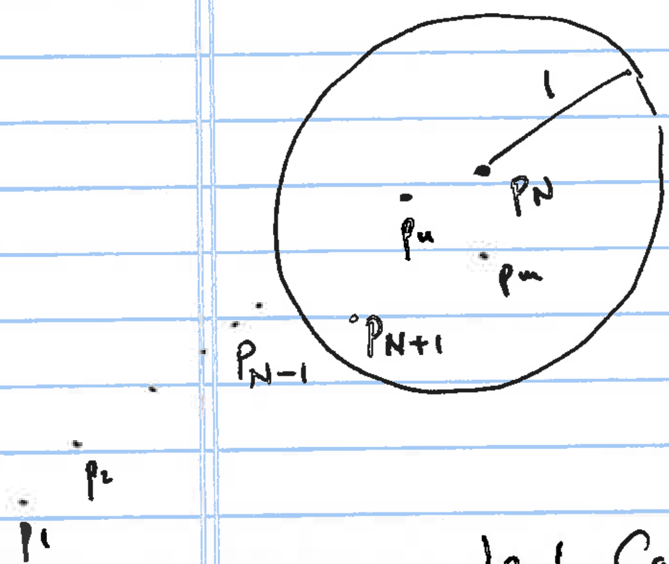
In $\mathbb{R}^k$

(c) Every Cauchy sequence is bounded:

For $\varepsilon = 1$ $\exists N,$ $\forall n, m \geq N$ $d(p_m, p_n) < 1$.

$\Rightarrow$ $d(p_N, p_n) < 1$

$\Rightarrow$ $d(0, p_n) < 1 + |p_N|$



$R = \max\{|p_1|, |p_2|, |p_3|, \ldots |p_{N-1}|, |p_N|+1\}$

$\forall n$ $p_n \in \underbrace{\overline{N_R(0)}}_{\text{compact by Heine-Borel}} = \overline{B_R(0)}$

$\{p_n\}$ Cauchy sequence in compact $\overline{N_R(0)}$

by part (b) $\{p_n\}$ converges. #