3.4 Thm

Let $\{p_n\}$ be a sequence in $\mathbb{R}^k$ (standard metric)

$$p_n = (a_{1,n}, a_{2,n}, a_{3,n}, \ldots a_{k,n})$$

$$p = (a_1, a_2, \ldots a_k)$$

$$\left.\begin{matrix} p_n \to p \\ \text{in } \mathbb{R}^k \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} \lim_{n\to\infty} a_{i,n} = a_i \\ \text{in } \mathbb{R} \end{matrix}\right. \quad \text{for } i = 1, 2, 3, \ldots k.$$

Proof.

$(\Rightarrow:)$ $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N,\ |p_n - p| < \varepsilon.$

$$\varepsilon^2 > |p_n - p|^2 = \sum_{j=1}^{k} (a_{j,n} - a_j)^2 \geq |a_{j,n} - a_j|^2 \quad \forall j$$

$$\varepsilon > |a_{j,n} - a_j| \qquad \forall j = 1, \ldots k$$

$$\lim_{n\to\infty} a_{j,n} = a_j \qquad \forall j = 1, \ldots k$$

$(\Leftarrow:)$ $\forall j\ \forall \varepsilon > 0\ \exists N_j \in \mathbb{N}\ \forall n \geq N_j\ \ |a_{j,n} - a_j| < \frac{\varepsilon}{\sqrt{k}}.$

Let $N = \max\{N_1, N_2, \ldots N_k\}$

$\forall n \geq N \geq N_j$

$$|p_n - p|^2 = \sum_{j=1}^{k} (a_{j,n} - a_j)^2 < \sum_{j=1}^{k} \left(\frac{\varepsilon}{\sqrt{k}}\right)^2 = \sum_{j=1}^{k} \frac{\varepsilon^2}{k} = \varepsilon^2$$

$$|p_n - p| < \varepsilon$$

$$\lim_{n\to\infty} p_n = p.$$

Subsequence

Defn Given a sequence $\{p_n\}$, for any increasing sequence of natural numbers $n_1 < n_2 < n_3 < n_4 \cdots < n_k < n_{k+1} < \cdots$

We call $\{p_{n_k}\}$ a subsequence of $\{p_n\}$

Ex ①

p_n : 1, 2, 3, 4, 5, 6, ...

p_{n_k} : 1, 4, 9, 16, 25

$n_k = k^2$

② p_n 1, 0, 1, 0, 1, 0,

q_n 0, 1, 0, 1, 0, 1,

p_n is a subsequence of q_n, and

q_n is " " of p_n

but not the same sequence

(3770) **Prop** : (1) $p_n \to p$ in (X, d) $\iff$ Every subsequence $\{p_{n_k}\}$ of $\{p_n\}$, $\lim_{k\to\infty} p_{n_k} = p$. in (X, d)

proofs HW

(2) $\{p_n\}$ is a convergent sequence in (X, d) $\iff$ Every subsequence is convergent in (X, d)

3.6 **Thm:**

(a) Let $\{p_n\}$ be a sequence in a compact metric space (X, d). Then, there exists a a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ s.t.

$$\lim_{k\to\infty} p_{n_k} = p \in X.$$

(b) Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence (with standard metric).

Proof. (a) Let $E = \{p_n \mid n \in \mathbb{N}\}$ be the range of the sequence $\{p_n\}$.

Case 1 E is a finite set.

Then p_n must repeat a value $p \in E$ infinitely many times.

(otherwise finitely many p values attained by finitely many p_n would imply $\mathbb{N}$ is finite)

i.e. $\exists\ n_1 < n_2 < n_3 < n_4 < \dots < n_k < n_{k+1} < \dots$ for $k \in \mathbb{N}$ s.t.

$$p_{n_k} = p.$$

$$\lim_{k\to\infty} p_{n_k} = p.$$

(Ex) $p_n = (-1)^n$: $-1, 1, -1, 1, -1, \dots$

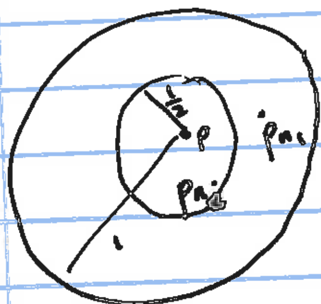
$\{p_n \mid n \in \mathbb{N}\} = \{-1, 1\}$ is a finite set of values.

Case 2 E is an infinite set

$E \subseteq (X, d)$ compact metric space

By Thm 2.37, E must have a limit pt., say p.

$$E = \{p_n \mid n \in \mathbb{N}\}, \quad p \in E'$$



Choose any n_1 s.t.

$$p_{n_1} \in \underbrace{N_1(p) \cap (E - \{p\})}_{\text{infinitely many elts.}} \neq \emptyset$$

Choose $n_2 > n_1$ s.t.

allows us

$$p_{n_2} \in \underbrace{N_{\frac{1}{2}}(p) \cap (E - \{p\})}_{\text{infinitely many elts}} \neq \emptyset$$

$\forall \ell \in \mathbb{N}$, Assuming p_{n_i} are chosen with $n_1 < n_2 < n_3 < \dots < n_\ell$, and $p_{n_\ell} \in N_{\frac{1}{\ell}}(p) \cap (E - \{p\})$. We choose $n_{\ell+1} > n_\ell$ s.t.

$$p_{n_{\ell+1}} \in \underbrace{N_{\frac{1}{\ell+1}}(p) \cap (E - \{p\})}_{\text{infinitely many elts : allows us to find } n_{\ell+1} > n_\ell} \neq \emptyset$$

So we have $d(p_{n_\ell}, p) < \frac{1}{\ell} \ \forall \ell$.

$$\Rightarrow p_{n_\ell} \to p, \text{ as } \ell \to \infty.$$

(b) Let $\{p_n\}$ be a bounded sequence in $\mathbb{R}^k$. (standard metric)

$$\exists R>0 \text{ s.t. } \forall n \quad p_n \in B_R(0) \subseteq \mathbb{R}^k$$

$$\overline{B_R(0)} = \{ x \in \mathbb{R}^k \mid |x| \le R \}$$

is closed & bounded in $\mathbb{R}^k$.

Heine-Borel $\Rightarrow$ $\overline{B_R(0)}$ is compact.

By part (a), $\{p_n\}$ a sequence in compact $\overline{B_R(0)}$. it must have a convergent subsequence.

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Sometimes this is referred as "Bolzano-Weierstrass for sequences".