Feb 14, 2018

To Conclude Chap II:

Connectedness:

Defn (i) Let $A, B \subseteq (X, d)$. A & B are said to be separated if $\left.\begin{array}{l}\bar{A} \cap B = \emptyset \\ A \cap \bar{B} = \emptyset\end{array}\right)$ $(\Rightarrow A \cap B = \emptyset)$

(ii) A subset $E \subseteq (X, d)$ is called connected if there doesn't exist any $A, B \subseteq X$ s.t. (i) $A \cup B = E$, $A \neq \emptyset$, $B \neq \emptyset$,
(ii) A & B are separated.

Caution: $(0,1), (1,2)$ are separated in $\mathbb{R}^1$. (open intervals)
$\underbrace{(0,1], (1,2)}_{\text{disjoint}}$ are not separated
$(0,1] \cap \overline{(1,2)} = \{1\}$.

(2.47) Thm: Let $E \subseteq \mathbb{R}^1$.
E is connected $\Longleftrightarrow$ $\forall x, y \in E \; (\forall z \in \mathbb{R} \;\; x < z < y \Rightarrow z \in E)$

Proof

$(\Rightarrow)$ Assume E is connected
To prove $\forall z \in \mathbb{R} \, (x < z < y \Rightarrow z \in E)$.
Suppose not $\forall z \in \mathbb{R} \, (x < z < y \Rightarrow z \in E)$
$\exists z \in \mathbb{R} \;\; x < z < y$ & $z \notin E$

Let $A = E \cap (-\infty, z)$
$B = E \cap (z, \infty)$

sub Lemma $\left[E \subseteq F \implies \bar{E} \subseteq \bar{F} \right]$

Proof $E \subseteq F$

$\bar{F} = F \cup F'$ (defn of closure)

$E \subseteq \bar{F}$

$\bar{F}$ closed (2.27a)

$E \subseteq \bar{F}$ closed

$\bar{E} \subseteq \bar{F}$ by (2.27 c)

Continue Proof of Thm 2.47 ($\implies$:)

$$A = E \cap (-\infty, z)$$

$$A \subseteq (-\infty, z)$$

$$\bar{A} \subseteq \overline{(-\infty, z)} = (-\infty, z].$$

$$B = E \cap (z, \infty) \subseteq (z, \infty)$$

$$\bar{A} \cap B \subseteq (-\infty, z] \cap (z, \infty) = \emptyset.$$

Similarly $A \cap \bar{B} \subseteq (-\infty, z) \cap [z, \infty) = \emptyset$.

Hence we obtain a separation of E, which is connected

Contradiction

Hence $\forall z \in \mathbb{R}$ $(x < z < y \implies z \in E)$.

Thm 2.47

$(\Leftarrow:)$ Assume $\forall x, y \in E \ (\forall z \in \mathbb{R},\ x < z < y \Rightarrow z \in E)$ (*)

To prove E is connected.

Proof by using contradiction

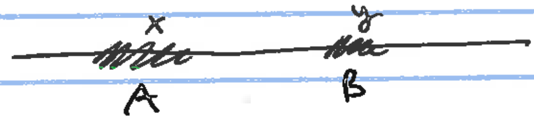
Suppose E is not connected,

$\exists$ a separation

$\exists A, B \subseteq X \quad A \neq \phi,\ B \neq \phi$

$E = A \cup B$

$\bar{A} \cap B = \phi,\ A \cap \bar{B} = \phi.$

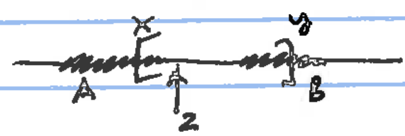


$\exists x \in A \subseteq E$

$\exists y \in B \subseteq E$

$(x < y$ or $x > y)$

Case 1 $\ x < y$ (Case 2 : $x > y$, same proof) just interchange $x \leftrightarrow y$, $A \leftrightarrow B$.



$[x, y] \cap A \neq \phi$ contains x

$[x, y] \cap A$ bdd

LUB property $\Rightarrow \exists \sup([x, y] \cap A) = z \in \mathbb{R}.$

$z \in \overline{[x, y] \cap A} \subseteq \bar{A}$, $z \notin B$ (since $\bar{A} \cap B = \phi$)

$x \leq z < y$ $\quad (y \in B)$

Sub Case i $\ z \notin A$

$z \notin A,\ z \notin B \Rightarrow z \notin E$, and $x < z < y$ contradicts (*) above

Sub Case (ii) $\ z \in A$

$(A \cap \bar{B} = \phi \Rightarrow)\ z \notin \bar{B}$ which is a closed set, & its complement is open.

$\exists r > 0 \left\{ \begin{array}{l} (z - r, z + r) \cap \bar{B} = \phi. \\ \text{and} \quad z + r < y \end{array} \right.$

$\left. \begin{array}{l} x < z + \frac{r}{2} < y \\ z + \frac{r}{2} \notin A \quad z + \frac{r}{2} \notin B \end{array} \right\}$ contradicts (*)

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Chap III:

Defn A sequence $\{p_n\}$ in (X, d) is said to converge in X if

$$\exists p \in X, \ \forall \varepsilon > 0 \ \exists N \ \forall n > N \ d(p_n, p) < \varepsilon.$$

Notation $\lim_{n\to\infty} p_n = p$ or $p_n \to p$.

- $\{p_n\}$ is said to diverge if it doesn't converge to any p in X
- $\{p_n\}$ is called bounded if the range $\{p_n \mid n \in \mathbb{N}\}$ is a bounded set.

i.e. $\exists p_0 \in X \ \exists R > 0$ s.t. $\forall n \in \mathbb{N}$ $\left\{ \begin{array}{l} d(p_0, p_n) < R. \\ \text{i.e. } p_n \in N_R(p_0) \end{array} \right.$

Thm: Let $\{p_n\}$ be a sequence in (X, d)

(a) $p_n \to p \iff \forall r$, $N_r(p)$ contains all but finitely many p_n.

(b) $p_n \to p$, $p_n \to p' \implies p = p'$

(c) $p_n \to p$ then $\{p_n\}$ is bounded.

[Ex $p_n = (-1)^n$ is bdd but not convergent in $(\mathbb{R}, \|\cdot\|)$

(d) $E \subseteq X$, $p \in E'$

$\implies \exists \{p_n\}$ in E s.t. $p_n \to p$.

proofs of (a), (b), (c) HW to read

Proof of (d)

$\forall n \in \mathbb{N}$ $\quad N_{\frac{1}{n}}(p) \cap (E - \{p\}) \neq \phi$. since $p \in E'$.

Choose any $p_n \in N_{\frac{1}{n}}(p) \cap (E - \{p\}) \neq \phi$

so that $d(p, p_n) < \frac{1}{n}$.

$\forall \varepsilon > 0$ choose $N = \frac{1}{\varepsilon}$ if $n > N$ then

$$d(p_n, p) < \frac{1}{n} < \frac{1}{N} = \varepsilon .$$

HW to read Thm 3.3 / and its proofs.

3.4 Thm: Let $\{p_n\}$ be a sequence in $\mathbb{R}^k$.

$$p_n = (a_{1,n}, a_{2,n}, a_{3,n}, \dots a_{k,n})$$

$$p_n \to p = (a_1, a_2, \dots a_n) \iff \lim_{n \to \infty} a_{i,n} = a_i \quad \forall i = 1, 2, \dots k.$$

Proof on Friday.