Feb 12, 2018

①

(2.41) <u>Thm:</u> The following are equivalent for subsets $E \subseteq \mathbb{R}^k$.

(a) E is closed and bounded

(b) E is compact

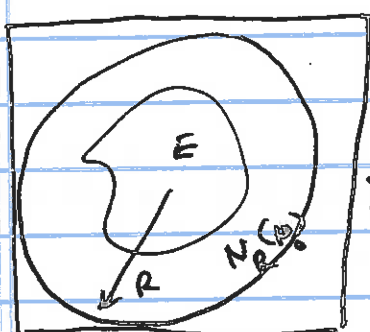
(c) Every infinite subset S of E has a limit pt in E.

<u>Proof.</u>

(a) $\Rightarrow$ (b)

E is closed & bounded.

E bdd $\Rightarrow$ $E \subseteq N_R(p)$



$\exists$ cell I s.t. $E \subseteq N_R(p) \subseteq I$

I is compact (Thm 2.40)

E closed subset of compact I

$\Rightarrow E$ is compact Thm 2.35

(b) $\Rightarrow$ (c) Thm 2.37 (Did on Friday 2/9/18)

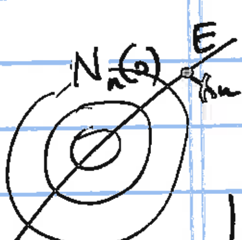
(c) $\Rightarrow$ (a)

<u>Assume:</u> Every infinite subset S of E has a limit pt in E.

<u>To prove:</u> E is bounded and closed

(i) Suppose E is not bounded

$\forall n \in \mathbb{N} \quad E \not\subseteq N_n(0)$



$\forall n \in \mathbb{N}, \exists x_n \in E \quad |x_n - 0| \geq n$

Let $S = \{x_n \mid n \in \mathbb{N}\}$.

If S were a finite set (i.e. x_n repeated some values as n changed.)

then $\{|x_n| \mid n \in \mathbb{N}\}$ would have a largest value, ℓ

Then we choose $m \in \mathbb{N}$ s.t. $m > \ell$, (Arch.P.

to obtain a contradiction: $m \leq |x_m| \leq \ell < m$

S is an infinite set, $\subseteq E$.

$\exists\, q \in S' \cap E$. (hypothesis: (c))

$N_1(q)$ would have infinitely many pts x_{n_ℓ} of S

$\forall x_{n_\ell} \in N_1(q) \cap S$ $\quad |x_{n_\ell} - q| < 1$ $\Big\downarrow$ Triangle Ineq.

$$\underset{\infty}{\underset{\downarrow}{n_\ell}} < |x_{n_\ell}| < \underbrace{|q| + 1}_{\text{fixed value}}.$$

Contradiction.

Hence E is bounded.

(ii) To show E is closed.

Suppose not. i.e. $E' \not\subseteq E$

$\exists\, p_0 \in E'$, $p_0 \notin E$

$\forall n \in \mathbb{N}$ $\quad N_{\frac{1}{n}}(p_0) \cap (E - \{p_0\}) \neq \phi$.

$\exists\, z_n \in N_{\frac{1}{n}}(p_0) \cap (E - \{p_0\})$

$z_n \in E$, $\quad 0 < |z_n - p_0| < \frac{1}{n}$

Let $S_1 = \{ z_n \mid n \in \mathbb{N} \}$

Suppose S_1 is finite (i.e. z_n repeated values) as n changed.

$|z_n - p_0|$ would repeat values $\underbrace{c_1, c_2 \ldots c_\ell}_{\text{finitely many}} > 0$

$c = \min(c_1, c_2 \ldots c_\ell) > 0$

$c \leq |z_n - p_0| < \frac{1}{n}$ $\forall n \in \mathbb{N}$

Contradicts Archimedean Principle.

Hence S_1 is an infinite set.

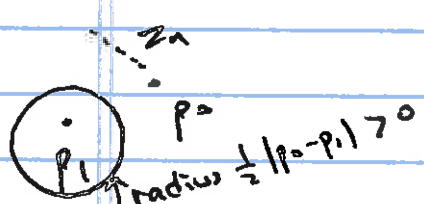
Apply hypothesis (c), to S_1.

S_1 has a limit pt in E. (by (c))

As chosen: $p_0 \notin E$ $\left.\begin{array}{l} p_0 \in E', \\ p_0 \in S_1' \end{array}\right\}$ but are there any other limit pts of S_1?

Is there another limit pt p_1 of S_1, s.t. $p_1 \in E$?

$p_1 \neq p_0$ $\quad |p_1 - z_n| \geq |p_1 - p_0| - |p_0 - z_n|$ (reverse triangle ineq.)



$\geq |p_1 - p_0| - \frac{1}{n} \geq \frac{1}{2}|p_1 - p_0| > 0$

for $n > \left(\frac{1}{2}|p_0 - p_1|\right)^{-1}$

There are only finitely many z_n in $N_{\frac{1}{2}|p_0 - p_1|}(p_1)$.

$p_1 \notin S'$.

Conclusion: p_0 is the only pt of S_1', $p_0 \notin E$.

Gives a contradiction with (c). Hence $p_0 \in E \cap S_1'$. E is closed.

<u>Corollary</u>: Bolzano-Weierstrass

Every bounded infinite subset of $\mathbb{R}^k$ must have a limit pt.

Let (a) E Closed & bounded

(b) E compact

(c) Every infinite subset S of E must have a limit point in E, that is $S' \cap E \neq \emptyset$.

In $\mathbb{R}^k$ $\quad \underbrace{(a) \Longleftrightarrow (b)}_{\text{Heine-Borel Thm.}} \Longleftrightarrow (c)$

In all metric spaces $(c) \Longleftrightarrow (b) \Longrightarrow (a)$, $(b) \not\Longleftarrow (a)$

i.e. Closed & bounded subsets need not be compact in general metric spaces.

In More general context: In some Topological spaces / not metrizable

compact (b) $\underset{\not\Longleftarrow}{\overset{\text{(Hausdorff)}}{\Longrightarrow}}$ closed $\quad$ (Boundedness is defined by use of a metric)

(b) $\Downarrow$, $\not\Uparrow$ (c)