Corollary of Thm 2.36

Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of non-empty compact sets s.t. $\phi \neq K_{n+1} \subseteq K_n$ $\forall n$. Then

$$\bigcap_{n=1}^{\infty} K_n \neq \phi.$$

Proof: $K_{n_1} \cap K_{n_2} \cap \cdots \cap K_{n_\ell} = K_N \neq \phi.$

$N = \max n_1, \cdots n_\ell$

2.57 Thm: Let E be an infinite subset of a compact set K, ($\subseteq (X, d)$ a metric space). Then E must have at least one limit pt in K.

Proof: Suppose not, that is $E' \cap K = \phi$.

$\forall q \in K$, $q \notin E'$

not ($\forall r > 0$ $N_r(q) \cap (E - \{q\}) \neq \phi$)

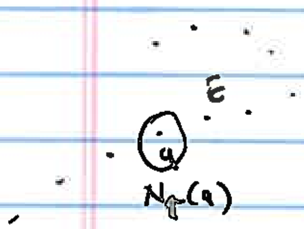
$\exists r > 0$ $N_r(q) \cap (E - \{q\}) = \phi$.

$N_r(q) \cap E \subseteq \{q\}$

define $V_q = N_r(q)$.

Since E is infinite,

there is no finite subcollection of $\{V_q \mid q \in K\}$ which covers $E \subseteq K$.



There is no finite subcollection of the open cover $\{V_q \mid q \in K\}$ of K, which covers K.

$\Rightarrow$ K is not compact. Contradicting Hypo.

Hence $E' \cap K \neq \phi$. #.

Compactness in $\mathbb{R}^k$:

2.38 Prop Let $I_n = [a_n, b_n]$ be a sequence of closed non-empty intervals in $\mathbb{R}$, s.t. $I_{n+1} \subseteq I_n$.

Then $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

Proof: $(I_n \neq \phi \Rightarrow a_n \leq b_n) \forall n$

$[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \Rightarrow$

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n.$$

$\Rightarrow \forall n, m \in \mathbb{N} \quad a_n \leq a_{n+m} \leq b_{n+m} \leq b_m$

Let $E = \{a_n \mid n \in \mathbb{N}\}$, $E \neq \phi$

E bdd above by $b_1, b_2, \dots, b_m \dots$

LUB property $\Rightarrow$ $\sup E$ exist

Let $A = \sup E$.

$\forall n \quad a_n \leq A$ — A is an upper bdd. for E

$\forall m \quad A \leq b_m$ — A is the least upper bd. for E

$A \in [a_n, b_n] \ \forall n$.

$$A \in \bigcap_{n=1}^{\infty} [a_n, b_n] \neq \phi.$$

2.39 **Prop:** Let $k \in \mathbb{N}$, Let $\{I_n\}$ be a sequence of non-empty k-cells in $\mathbb{R}^k$ s.t.

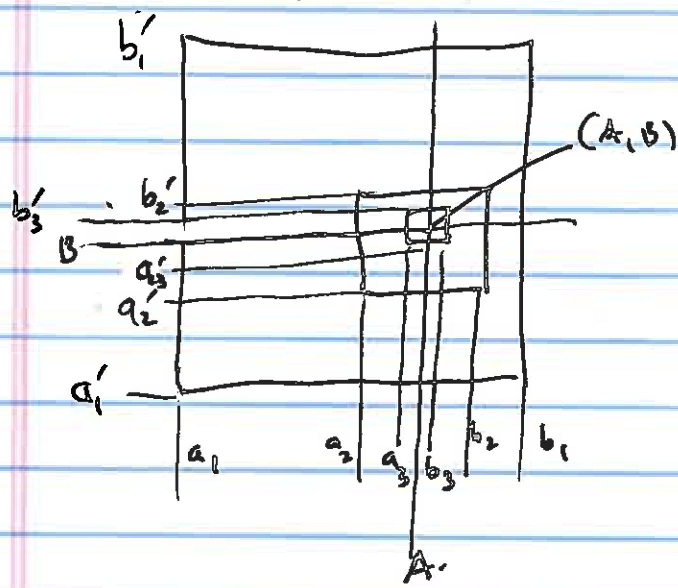
$$I_{n+1} \subseteq I_n \quad \forall n.$$

$$\text{Then } \bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof (HW to read) p 39

Recall k-cell:

$[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k]$.



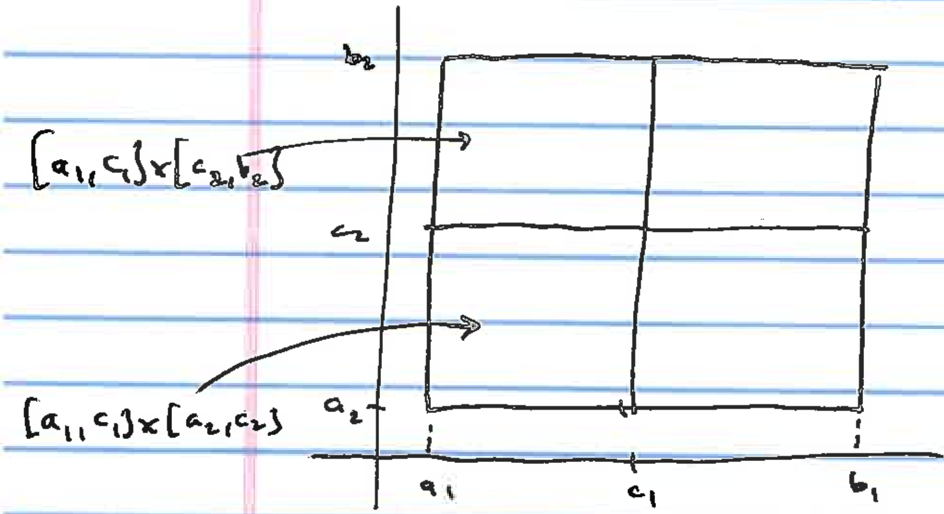
Thm: Every k-cell is compact in $\mathbb{R}^k$.

$I \neq \phi.$

Proof:

$$I = \{(x_1, x_2, \dots x_k) \mid a_i \le x_i \le b_i, \forall i = 1,2,3,\dots k\}$$

where $a_i \le b_i \; \forall i$



$$\delta = \left(\sum_{i=1}^{k} (a_i - b_i)^2\right)^{\frac{1}{2}} = \text{diam}(I)$$

$\forall p, q \in I$

$$\|p - q\| \le \delta$$

(WTS) I is compact.

Let $\{G_\alpha \mid \alpha \in A\} = \mathscr{C}$ be an open cover of I.

Suppose that I cannot be covered by any finite subcover of $\mathscr{C}$, (that is I is not compact,

Let $c_j = \dfrac{a_j + b_j}{2} \quad \forall j$

Divide I into 2^k subcells Q_i by taking products of either ($[a_j, c_j]$ or $[c_j, b_j]$) for each j.

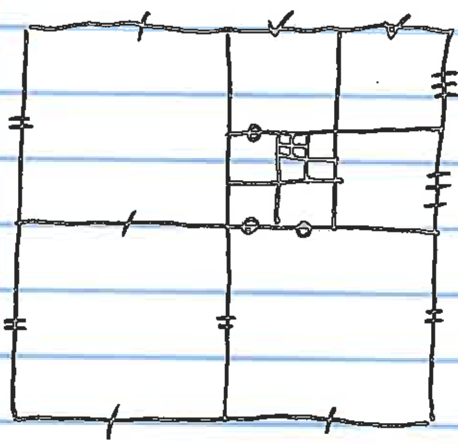
e.g. $\underbrace{[a_1, c_1] \times [a_2, c_2] \times [c_3, b_3] \times [c_4, b_4] \times \cdots \times [a_k, c_k]}_{2^k \text{ such choices.}}$

If we can cover each Q_i with finitely many of G_α, and since there are finitely many Q_i, (2
then we can cover the cell I with finitely many G_α

However, we cannot cover I with finitely many G_α. So at least one of the Q_i's can't be covered with finitely many G_α.

Let I_1 be any of these subcells which cannot be covered by finitely many G_α. $\operatorname{diam}(I_1) = \frac{\delta}{2}$.

Next to subdivide I_1 into 2^k subcells, by dividing each side length into 2 equal pieces. One of those 2^k subcells cannot be covered with finitely many G_α.



Proceed inductively

- $I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$
- $\operatorname{diam}(I_n) = \frac{\delta}{2^n}$
- I_n can't be covered with finitely many of G_α's. ⊛

Then: $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. (by Thm 2.39)

Let $p_0 \in \bigcap_{n=1}^{\infty} I_n$.

$p_0 \in I$, covered by $\{G_\alpha \mid \alpha \in A\}$

$p_0 \in G_{\alpha_0}$ for some α_0.

G_{α_0} is open.

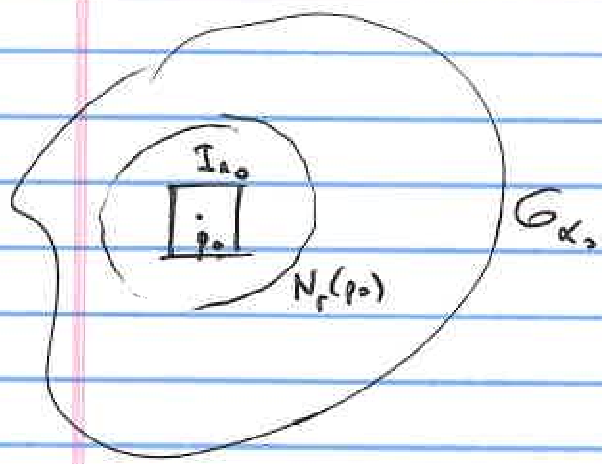
$\exists r > 0 \quad N_r(p_0) \subseteq G_{\alpha_0}$.

$\exists n_0 \in \mathbb{N}$ s.t. $\frac{\delta}{2^{n_0}} < r$.

$p_0 \in \bigcap_{n=1}^{\infty} I_n \implies p_0 \in I_{n_0}$,

$\forall q \in I_{n_0} \quad \| q - p_0 \| \leq \operatorname{diam} I_{n_0} = \frac{\delta}{2^{n_0}} < r$

$I_{n_0} \subseteq N_r(p_0) \subseteq G_{\alpha_0}$.



I_{n_0} can be covered with finitely many G_α, namely G_{α_0}.

Contradiction, with (*) p.5

$\implies I$ is compact.

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