Compactness: (X, d) metric space

Defn Let (X,d) be a metric space, $E \subseteq X$.
A collection $\{G_\alpha \mid \alpha \in A\}$ of open subsets of X is called an open cover for E if $E \subseteq \bigcup_{\alpha \in A} G_\alpha$.

Def: A subset K of (X,d) is called compact if every open cover of K has a finite subcover, i.e.

$$K \subseteq \bigcup_{\alpha \in A} G_\alpha, \quad \text{all } G_\alpha \text{ open}$$

$$\Rightarrow \exists\, \alpha_1, \alpha_2, \dots \alpha_\ell \text{ s.t. } K \subseteq \bigcup_{i=1}^{\ell} G_{\alpha_i}$$

Ex 1. Every finite set is compact.
2. $\emptyset$ is compact
3. $(\mathbb{R}, \|\cdot\|)$ is not compact:

$$\mathbb{R} = \bigcup_{n=1}^{\infty} \underset{-n<x<n}{(-n,n)}, \quad \text{but for all finite subcollections}$$

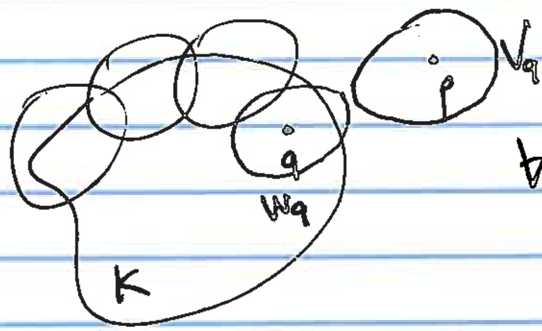
$$\bigcup_{i=1}^{\ell} (-n_i, n_i) = (-N, N) \neq \mathbb{R}.$$

where $N = \max(n_1, n_2, \dots n_\ell)$

2.34 **Thm:** Compact subsets K of metric spaces are closed.

Proof: $K \subseteq X$ WTS K^c is open.

WTS $\forall p \in K^c\ \exists r>0\ \underbrace{N_r(p) \subseteq K^c}_{N_r(p) \cap K = \phi}$.



$\forall q \in K$

$$\left. \begin{array}{l} q \in W_q = N_{\frac{1}{2}|p-q|}(q) \\ V_q = N_{\frac{1}{2}|p-q|}(p) \end{array} \right\} W_q \cap V_q = \phi.$$

$$K \subseteq \bigcup_{q \in K} W_q \xRightarrow{K \text{ compact}} \exists\ q_1, q_2, \ldots, q_\ell \text{ s.t. } K \subseteq \bigcup_{i=1}^{\ell} W_{q_i}$$

Let $r = \min_{1 \le i \le \ell} \frac{1}{2}|p-q_i| > 0$

Let $V = B_r(p) \subseteq B_{\frac{1}{2}|p-q_i|}(p) = V_{q_i}$

$$V \cap W_{q_i} \subseteq V_{q_i} \cap W_{q_i} = \phi.$$

$$V \cap K = V \cap \left(\bigcup_{i=1}^{\ell} W_{q_i} \right) = \bigcup_{i=1}^{\ell} (V \cap W_{q_i}) = \phi.$$

$p \in V = B_r(p) \subseteq K^c$. #

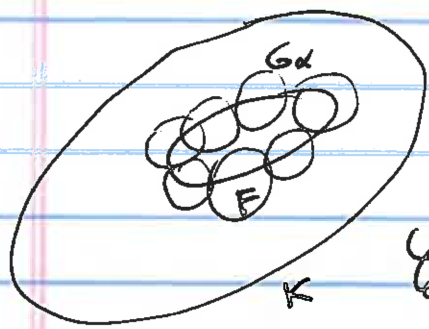
Remark: Same proof will work if For a given metric space, $\left\{ \begin{array}{l} \text{All } |p-q| \text{ are replaced by } d(p,q) \\ \text{All } B_r(p) \text{ " " } N_r(p). \end{array} \right.$

2.35 Thm. In a metric space, closed subsets of compact sets are compact

i.e. $F \subseteq K \subseteq X$ metric space

F closed, K compact $\Rightarrow$ F compact

Proof



Let $\{G_\alpha \mid \alpha \in A\}$ be an open cover of F.

F closed $\Rightarrow$ F^c is open.

$\mathcal{G}_0 = \{F^c\} \cup \{G_\alpha \mid \alpha \in A\}$ is an open cover of K

K compact $\Rightarrow$ $\exists$ a finite subcollection $\mathcal{G}$ of $\mathcal{G}_0$ s.t

$\bigcup_{G \in \mathcal{G}} G$ covers K; i.e $\mathcal{G}$ covers K.

if $F^c \in \mathcal{G}$, then take it away.

$\mathcal{G}_1 = \mathcal{G} - \{F^c\}$ is a finite collection of open sets

$\mathcal{G}_1$ covers F since $\mathcal{G}$ covers K and F^c doesn't cover any part of F

$\mathcal{G}_1 \subseteq \{G_\alpha \mid \alpha \in A\}$.

#

Corollary: F closed, K compact $\Rightarrow$ $F \cap K$ is compact.

(Not in the book)

Prop: Every compact set K in a metric space is bounded

Proof Choose a point $p_0 \in X$

$$\forall p \in X \quad d(p_0, p) < \infty$$
$$\exists n \in \mathbb{N} \text{ s.t. } d(p_0, p) < n.$$

$$p \in N_n(p_0)$$

$$\bigcup_{\substack{n=1 \\ n \in \mathbb{N}}}^{\infty} N_n(p_0) = X$$

$$K \subseteq \bigcup_{n=1}^{\infty} N_n(p_0), \quad K \text{ compact.}$$

$$\exists\, n_1, n_2, \ldots n_\ell \in \mathbb{N} \text{ s.t. } K \subseteq \bigcup_{i=1}^{\ell} N_{n_i}(p_0) = N_R(p_0)$$

$$R = \max_{1 \le i \le \ell} n_1, n_2, \ldots n_\ell$$

For K $\exists R > 0$ s.t. $K \subseteq N_R(p_0)$.

K is bounded.

Corollary: In every metric space, every compact set is closed and bounded.

Converse is true in $\mathbb{R}^k$. (Heine-Borel Thm).
Converse is false in general.

To be proved

Ex In $\mathbb{R}^k$: Finite sets are compact

In $\mathbb{R}^k$: $\overline{N_R(p_0)}$ is compact $\Longleftarrow$ Heine-Borel Thm

In $\mathbb{R}^3$: $[1,2]\times[3,5]\times[-1,0]$ compact in $\mathbb{R}^3$

2.36 **Thm**: Let $\{K_\alpha \mid \alpha \in A\}$ be a collection of compact subsets of (X, d).

If $\{K_\alpha \mid \alpha \in A\}$ satisfies finite intersection property (f.i.p.)

that is: for any $\alpha_1, \alpha_2 \ldots \alpha_\ell \in A$ $\bigcap_{i=1}^{\ell} K_{\alpha_i} \neq \phi$,

then $\bigcap_{\alpha\in A} K_\alpha \neq \phi$.

Ex $\bigcap_{n=1}^{\infty} [n, \infty) = \phi$ (*)

$\bigcap_{n_1, n_2 \ldots n_\ell} [n_i, \infty) = [M, \infty) \neq \phi$

$M = \max\{n_1, n_2, \ldots, n_\ell\}$

closed but not compact

$[n, \infty)$ satisfy f.i.p. but overall intersection is ϕ (*)

Proof of (2.36)

$\{K_\alpha \mid \alpha \in A\}$ satisfies f.i.p.

Suppose $\bigcap_{\alpha \in A} K_\alpha = \phi$.

Choose any K_1 in $\{K_\alpha \mid \alpha \in A\}$ (and rename it K_1) $A' = A - \{1\}$.

$$K_1 \cap \bigcap_{\substack{\alpha \neq 1 \\ \alpha \in A}} K_\alpha = \phi$$

$$K_1 \subseteq \Big(\bigcap_{\substack{\alpha \neq 1 \\ \alpha \in A}} K_\alpha\Big)^c = \bigcup_{\substack{\alpha \neq 1 \\ \alpha \in A}} K_\alpha^c$$

open since K_α compact $\Rightarrow K_\alpha$ closed

$\{K_\alpha^c \mid \alpha \neq 1, \alpha \in A\}$ is an open cover of K_1, compact

$\Rightarrow \exists$ finite subcover : $\alpha_1, \alpha_2, \dots \alpha_\ell$ $\quad K_1 \subseteq \bigcup_{i=1}^{\ell} K_{\alpha_i}^c$

$$\phi = K_1 \cap \Big(\bigcup_{i=1}^{\ell} K_{\alpha_i}^c\Big)^c = \underbrace{K_1 \cap \bigcap_{i=1}^{\ell} K_{\alpha_i}}_{\text{finitely many } K_1, K_{\alpha_1}, \dots K_{\alpha_\ell}}$$

This contradicts f.i.p.. Hence $\bigcap_{\alpha \in A} K_\alpha \neq \phi$.