May 5, 2017

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Final Exam Thursday May 11
8 - 10 pm
61 SH

Office Hrs. W & Th. 10 - 12 noon.

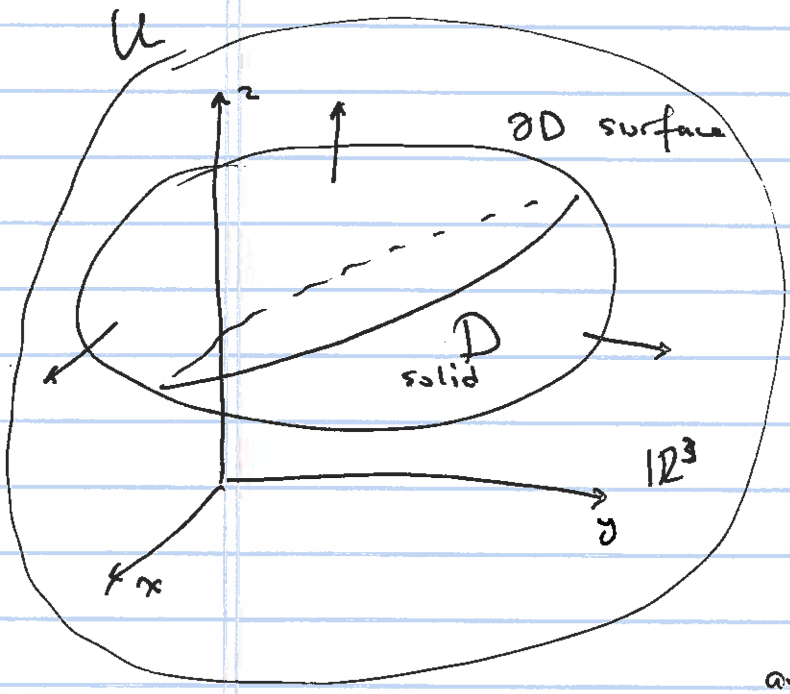
Review Session Tuesday May 9 5:30 - 7:30 pm
Room TBA

Practice questions + old Final posted on Course web page

Exam: questions are from Chap 5-7
responsible for all of Chap 1-7.

Can bring one page of your own notes.
$8\frac{1}{2}'' \times 11''$

7.3 Divergence Thm
(Gauss' Theorem.)



Let D be a bounded solid region in $\mathbb{R}^3$, whose boundary ∂D consists of finitely many piecewise smooth surfaces with outward orientation

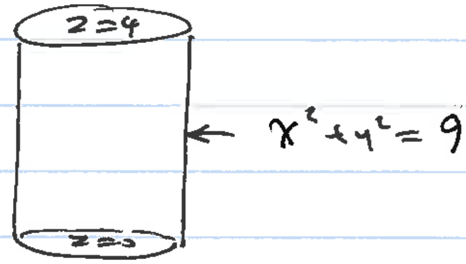
Let $F : U \overset{\text{open}}{\subseteq} \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field, and $D, \partial D \subseteq U$.

Then

$$\iiint_D \underbrace{\operatorname{div} F}_{\nabla \cdot F} \, dV = \iint_{\partial D} \vec{F} \cdot d\vec{S}$$

Redo

Exc #14 (7.2) p489, by using Gauss' Thm.



D : inside
$0 \le z \le 4$
$x^2+y^2 \le 9$
solid cylinder

$$S = \partial D$$

did it 5/4/17

Long calculation

$$72\pi = \iint_{\partial D = S} \vec{(x,y,0)} \cdot \vec{dS} \overset{\text{Gauss}}{=} \iiint_D \operatorname{div} F \cdot dV$$

$$\nabla \cdot (x,y,0) = \operatorname{div}(x,y,0) = \frac{\partial}{\partial x} \cdot x + \frac{\partial}{\partial y} \cdot y + \frac{\partial}{\partial z} \cdot 0 = 2.$$

$$\iiint_D 2\, dV = 2 \cdot \text{volume } D = 2 \cdot \pi r^2 \cdot h$$

$$= 2 \cdot \pi \cdot 9 \cdot 4 = 72\pi.$$

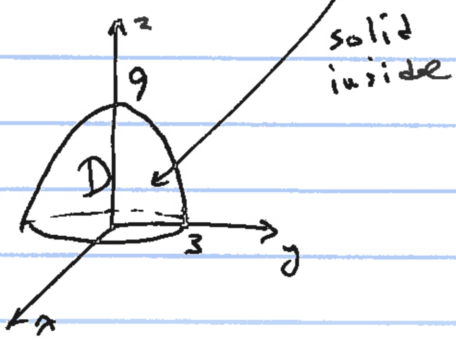
This is much shorter!

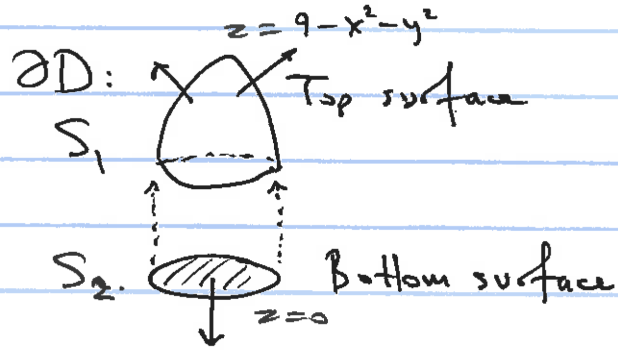
Exc #6 (7.3)

Verify Gauss' Thm $\iiint_D \operatorname{div} F\,dV = \iint_{\partial D} \vec{F}\cdot d\vec{S}$

$F = (x, y, z)$ $\operatorname{div} F = 3$

$D = \{(x,y,z) \mid 0 \le z \le 9 - x^2 - y^2\}$.





$$\iiint_D \operatorname{div} F\,dV = \iint_{\partial D} \vec{F}\cdot d\vec{S}$$

$$= \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$$\iiint_D \underbrace{\operatorname{div} F}_{3}\,dV = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} 3\,dz\,dy\,dx$$

Convert to cylindrical

$$= \int_0^{2\pi}\int_0^3\int_0^{9-r^2} 3\,r\,dz\,dr\,d\theta$$

Jacobian

$$= \int_0^{2\pi} \int_0^3 3rz \Big|_{z=0}^{z=9-r^2} dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 \underbrace{3r(9-r^2)}_{27r-3r^3} - 0 \quad dr\, d\theta$$

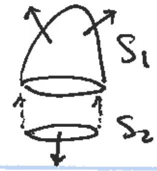
$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^3 27r - 3r^3\, dr\right)$$

$$= 2\pi \cdot \left(\frac{27}{2}r^2 - \frac{3r^4}{4}\Big|_0^3\right)$$

$$= 2\pi\left(\frac{27\cdot 9}{2} - \frac{3\cdot 81}{4}\right) =$$

$$\iiint_D \operatorname{div} F \cdot dV = \frac{243\pi}{2}.$$

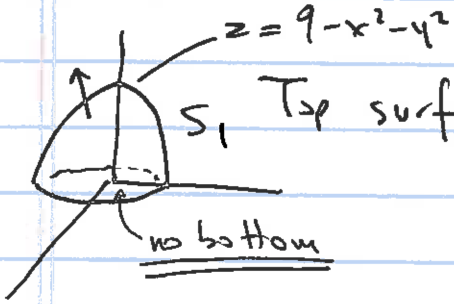
PTO for $\iint_{\partial D} \vec{F} \cdot d\vec{S}$



$$\partial D = S_1 \cup S_2$$

$$\iint_{S_1} F \cdot dS$$

$0 < s \leq 3$
$0 \leq t \leq 2\pi$



Top surface $\quad X = (s\cos t,\ s\sin t,\ 9 - s^2)$

$$X_s = (\cos t,\ \sin t,\ -2s)$$
$$X_t = (-s\sin t,\ s\cos t,\ 0)$$

$$N = (2s^2\cos t,\ 2s^2\sin t,\ s)$$

$s \geq 0$ upward normal

$$F(x,y,z) = (x,y,z)$$

$$\iint_{S_1} F \cdot dS = \int_0^{2\pi}\int_0^3 F(X(s,t)) \cdot N_X(s,t)\ ds\,dt$$

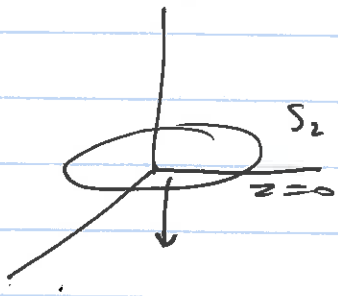
$$= \int_0^{2\pi}\int_0^3 (s\cos t,\ s\sin t,\ 9-s^2) \cdot (2s^2\cos t,\ 2s^2\sin t,\ s)\ ds\,dt$$

$$= \int_0^{2\pi}\int_0^3 \underbrace{2s^3\cos^2 t + 2s^3\sin^2 t}_{2s^3} + 9s - s^3\ ds\,dt$$

$$= \int_0^{2\pi}\int_0^3 (s^3 + 9s)\ ds\,dt$$

$$= 2\pi \cdot \left(\frac{s^4}{4} + \frac{9s^2}{2}\Big|_0^3\right) = 2\pi \cdot \left(\frac{81}{4} + \frac{81}{2}\right) = \frac{243\pi}{2}.$$

Bottom surface



$$Y(s,t) = (s,t,0)$$
$$Y_s = (1,0,0)$$
$$Y_t = (0,1,0)$$
$$N_Y = (0,0,1) \uparrow$$

$$\iint_{S_2} F\cdot dS = -\iint F(Y(s,t))\cdot N_Y(s,t)\,ds\,dt$$

$$\underbrace{(s,t,0)\cdot(0,0,1)}_{0}$$

$$= 0.$$

$$\iiint_D \operatorname{div} F\, dV = \frac{243\pi}{2}$$

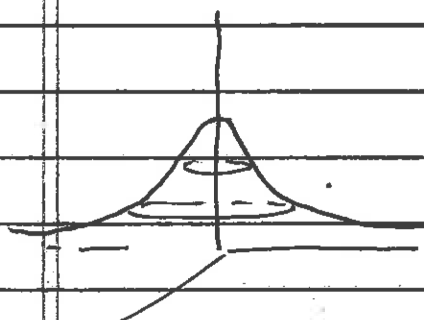
$$\underbrace{\iint_{S_1}\vec{F}\cdot d\vec{S}}_{\frac{243\pi}{2}} + \underbrace{\iint_{S_2}\vec{F}\cdot d\vec{S}}_{0} = \frac{243\pi}{2}$$

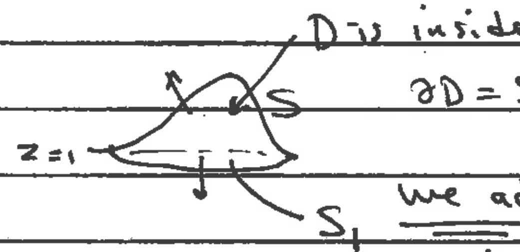
p 506

Exc #18 $\quad F = (x, y, 2-2z)$

S = graph of $\quad z = e^{1-x^2-y^2}$

above $z = 1$





$\partial D = S \cup S_1$

we add to close it up.

$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{want}} + \iint_{S_1} \vec{F} \cdot d\vec{S} \overset{\text{Gauss'}}{=} \underbrace{\iiint_D \operatorname{div} F \cdot dV}_{0} = 0$$

$\operatorname{div} F = 0$

S_1: $\quad \underline{X}(s,t) = (s, t, 1) \quad$ disk/circle on the plane $z = 1$

needed normal $N = (0, 0, -1)$; $\quad \cancel{N_{\underline{X}} = (0, 0, 1)}$

$$-\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint \underbrace{F(\vec{X}(s,t))}_{} \cdot \vec{N}_{\underline{X}} \cdot ds\,dt = 0$$

opposite normal

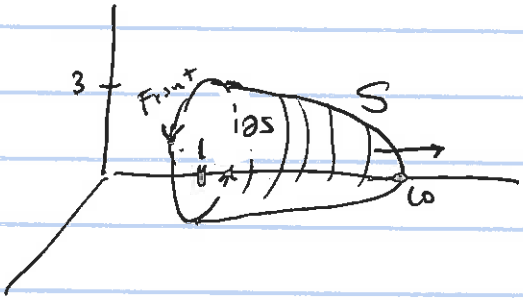
$$\underbrace{(s, t, 0) \cdot (0, 0, -1)}_{0}$$

$$\iint_S \vec{F} \cdot d\vec{S} = 0$$

Extra problem

7.3 Exc #11 (Double Stokes)

$y = 10 - x^2 - z^2 \qquad y \geq 1$



$F = (2xyz + 5z,\ e^x\cos yz,\ x^2y)$

curl F

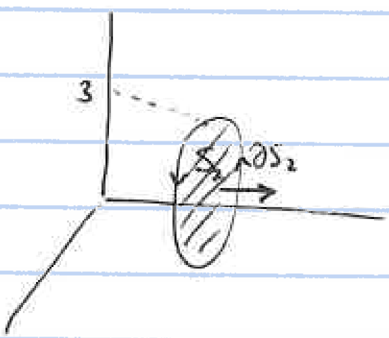
$= (x^2 + ze^x\sin yz,\ \cancel{2xy} + 5 - \cancel{2xy},\ e^x\cos yz - 2xz)$

F

$$\iint_S (\vec{\nabla} \times F) \cdot \vec{dS} = \oint_{\partial S} F \cdot ds$$

curl $F(\underline{X}(s,t))$ = curl $F(s\cos t,\ 10 - s^2,\ s\cdot\sin t) = (\ \cdots\ )$

$F(\vec{x}(t)) = F(\cos t,\ 1,\ \sin t) = \cdots$



$$\int_{\partial S = \partial S_2} F \cdot ds = \iint_{S_2} (\nabla \times F) \cdot dS$$

$Y(s,t) = (s, 1, t) \qquad s^2 + t^2 \leq 9$

$Y_s = (1, 0, 0)$

$Y_t = (0, 0, 1)$

$N = (0, -1, 0)$

$$\iint_{S_2} \nabla \times F\, dS = -\iint_{D = \{(s,t) \mid s^2 + t^2 \leq 9} (*, 5, *) \cdot (0, 1, 0)\, ds\, dt = -5 \cdot \pi 3^2 = -45\pi.$$