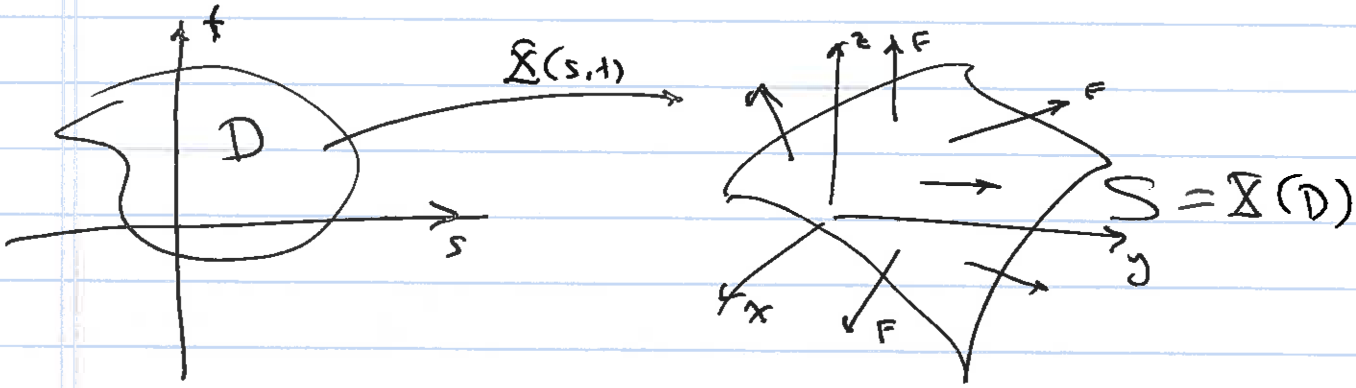
May 3, 2017

(*) Learn all examples done in class well!

7.2 FLUX Integrals

<u>Defn</u> Let $\underline{X}(s,t) : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$
be a 1-1 smooth parametrization,
$(N_{\underline{X}} \neq 0)$, of an oriented surface $S = \underline{X}(D)$



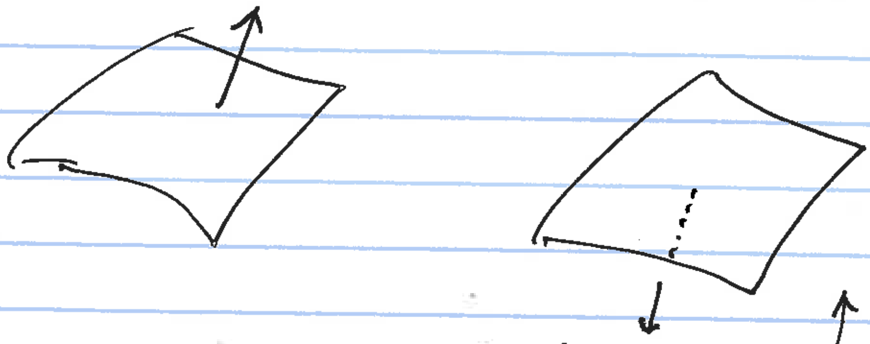
Let F be a vector field defined in a neighborhood of S
(Want F to be continuous)
Then one defines the flux of F thru S

$$\iint_S \vec{F} \cdot \vec{dS} = \iint_{\underline{X}} \vec{F} \cdot \vec{dS} =$$

$$= \iint_D \vec{F}(\underline{X}(s,t)) \cdot \underbrace{\frac{\partial \vec{\underline{X}}}{\partial s} \times \frac{\partial \vec{\underline{X}}}{\partial t}}_{\vec{N}} \, ds\, dt$$

<u>ORIENTATION</u> Curves: $\longrightarrow$ one chosen direction of the curve
$\longleftarrow$
There are two different orientations of a curve

ORIENTATION of a Surface is a choice of a side.



Each choice is an orientation.

$\mathbf{X}(s,t)$ given

$\dfrac{\partial \mathbf{X}}{\partial s}, \dfrac{\partial \mathbf{X}}{\partial t}$

$$N = \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t}$$

N determines a choice of a side for a given parametrization $\mathbf{X}$

$$dS = \|N\| \, ds\,dt$$

$$d\vec{S} = \vec{N} \cdot ds\,dt$$

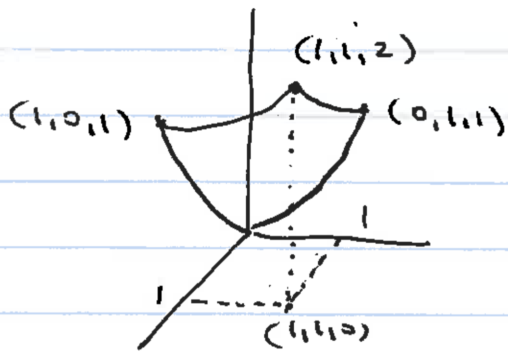
$$\vec{n} = \frac{\vec{N}}{\|\vec{N}\|} \quad \text{unit normal vector.}$$

$$d\vec{S} = \vec{n}\,dS = \frac{\vec{N}}{\cancel{\|\vec{N}\|}} \cdot \cancel{\|N\|}\, ds\,dt = \vec{N}\,ds\,dt$$

Ex. 1 let $\underline{X}(s,t) = (s,\ t,\ \underbrace{s^2+t^2})$, with $x = s$, $y = t$, $z = s^2+t^2$

$$\boxed{\begin{array}{l} 0 \le s \le 1 \\ 0 \le t \le 1 \end{array}}$$

The domain of $\underline{X}$ = parametrization domain of S



Let $F = (xz,\ yz,\ xy)$ given

Calculate

$$\iint_{\underline{X}} \vec{F} \cdot d\vec{S}.$$

Soln :

$$\frac{\partial \underline{X}}{\partial s} = (1,\ 0,\ 2s)$$

$$\frac{\partial \underline{X}}{\partial t} = (0,\ 1,\ 2t)$$

$$N = \frac{\partial \underline{X}}{\partial s} \times \frac{\partial \underline{X}}{\partial t} = (-2s,\ -2t,\ 1)$$

$$F(\underline{X}(s,t)) = (s(s^2+t^2),\ t(s^2+t^2),\ st)$$

integration domain

$$\iint_{\underline{X}} \vec{F} \cdot d\vec{S} \overset{\text{Def}^n}{=} \iint_{\substack{0 \le u \le 1 \\ 0 \le v \le 1}} \underbrace{(s^3+st^2,\ ts^2+t^3,\ st)}_{F(\underline{X}(s,t))} \cdot \underbrace{(-2s,\ -2t,\ 1)}_{\vec{N}}\, ds\, dt$$

$$= \int_0^1 \int_0^1 \left(-2s^4 \underbrace{- 2s^2t^2 - 2t^2s^2}_{-4s^2t^2} - 2t^4 + st\right) \underset{=}{ds}\, dt$$

$$= \int_0^1 -\frac{2s^5}{5} - \frac{4}{3}s^3t^2 - 2st^4 + \frac{s^2}{2}t \Big|_{s=0}^{s=1} dt$$

$$= \int_0^1 \left(-\frac{2}{5} - \frac{4}{3}t^2 - 2t^4 + \frac{t}{2}\right) dt$$

$$= -\frac{2}{5}t - \frac{4}{9}t^3 - \frac{2}{5}t^5 + \frac{t^2}{4} \Big|_0^1$$

$$= -\frac{2}{5} - \frac{4}{9} - \frac{2}{5} + \frac{1}{4} = -\frac{179}{180}$$

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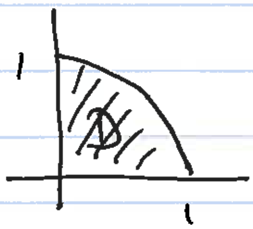
$$\mathbf{X} = (s+t,\ s-t,\ st)\ , \quad F(x,y,z) = (x,y,z)$$

$$D = \{(s,t) \mid s^2+t^2 \le 1,\ \ s \ge 0\ ,\ t \ge 0\}$$



Want $\displaystyle\iint_{\mathbf{X}} \vec{F} \cdot d\vec{S}$

$$\frac{\partial \mathbf{X}}{\partial s} = (1,1,t)$$

$$\frac{\partial \mathbf{X}}{\partial t} = (1,-1,s)$$

$$N = (s+t,\ t-s,\ -2)$$

$$= \iint_D \vec{F}(\mathbf{X}(s,t)) \cdot \vec{N}(s,t)\, ds\, dt$$

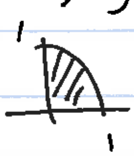
$F = \mathrm{Id}$

$$= \iint \underbrace{(s+t,\ s-t,\ st)}_{F(\mathbf{X}(s,t))} \cdot \underbrace{(s+t,\ t-s,\ -2)}_{N}\, ds\, dt$$

$D =$ 

$$= \iint \big(\underbrace{s^2+t^2+2st}_{(s+t)^2}\big) + \big(\underbrace{-s^2+2st-t^2}_{(s-t)(t-s)}\big) + (-2st)\ ds\,dt$$

$$= \iint 2st\ ds\,dt = \int_0^1 \int_0^{\sqrt{1-s^2}} 2st\ dt\ ds = \text{easy}$$



OR PTO

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$$\iint 2st\,ds\,dt \qquad \int_0^{\pi/2}\int_0^1 2(r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

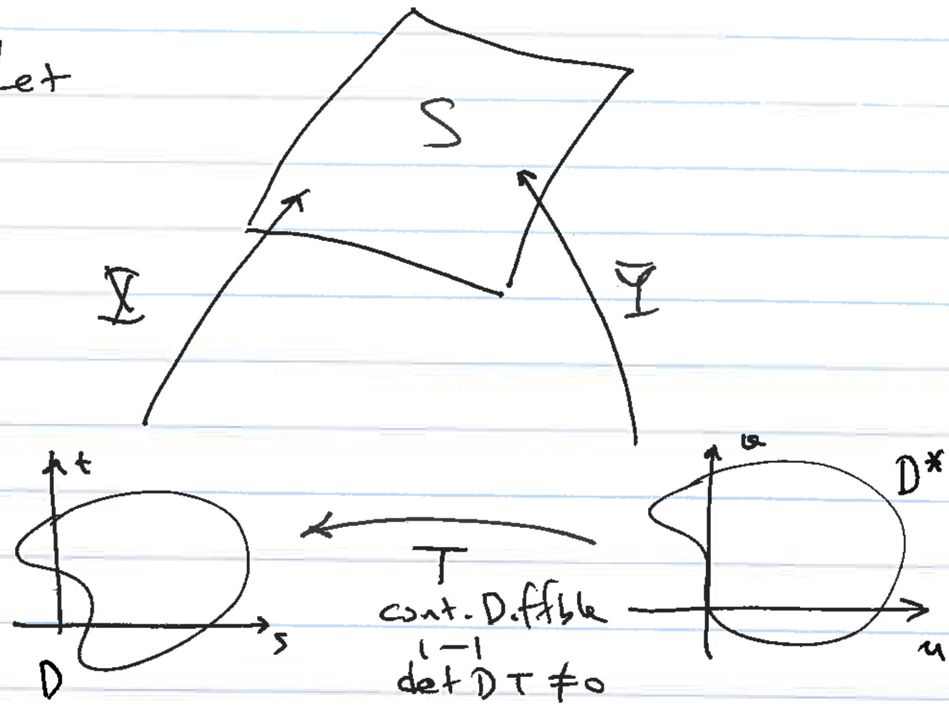
$$=\left(\int_0^1 r^3\,dr\right)\left(\int_0^{\pi/2} \underbrace{2\cos\theta\sin\theta}_{\sin 2\theta}\,d\theta\right)$$

$$=\frac{1}{4}\left(-\frac{\cos 2\theta}{2}\Bigg|_0^{\pi/2}\right)$$

$$=\frac{1}{4}\left(\frac{-\overbrace{\cos\pi}^{-1}+\overbrace{\cos 0}^{1}}{2}\right)=\frac{1}{4}$$

Equivalent Parametrizations:

Let



Let Both Φ & Ψ are smooth, 1-1 parametrizations of S
$N_{\Phi} \neq 0$, $N_{\Psi} \neq 0$, both onto S.

Let $T(u,v) = (s,t)$ and $\det(DT)$ have same sign over D^*.

Then
$$\Phi(T(u,v)) = \Psi$$

Φ & Ψ are Called equivalent parametrizations.

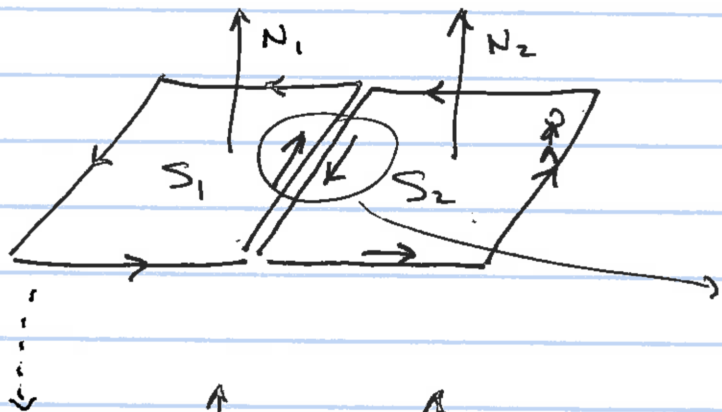
Furthermore:
$$\iint_{\Phi} \vec{F} \cdot d\vec{S} = \pm \iint_{\Psi} \vec{F} \cdot d\vec{S}$$

$N_{\Psi} = (\det DT) N_{\Phi}$
$\Longrightarrow N_{\Psi} \parallel N_{\Phi}$, but N_{Φ} and N_{Ψ} are:

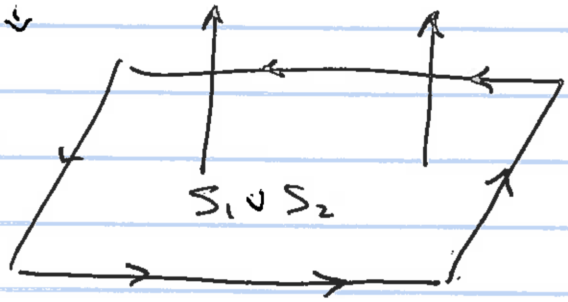
same orientation $\longleftrightarrow$ $+$ if $\det DT > 0$ $\longleftrightarrow$ in the same direction

opposite orientations $\longleftrightarrow$ $-$ if $\det DT < 0$ $\longleftrightarrow$ in the opposite directions

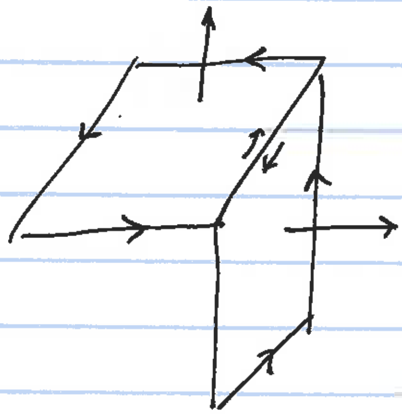
How to patch surfaces & their orientations:



Boundary integrals will cancel

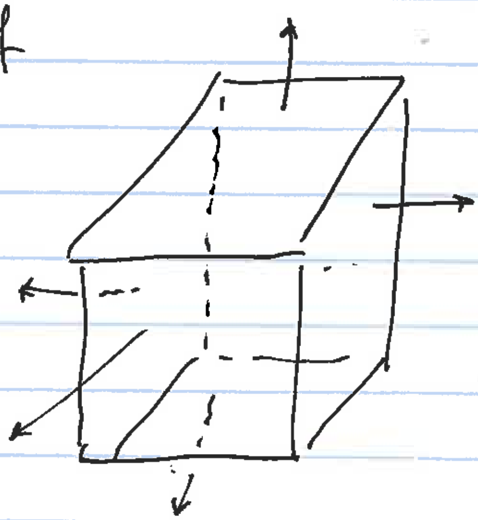


There is a continuous choice of a side (orientation)



There is a continuous choice of a side orientation

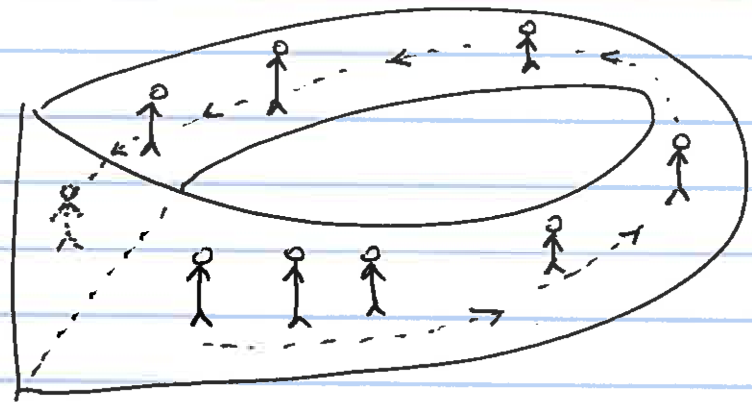
Boundary of a cube



There is a continuous choice of a side outward orientation.

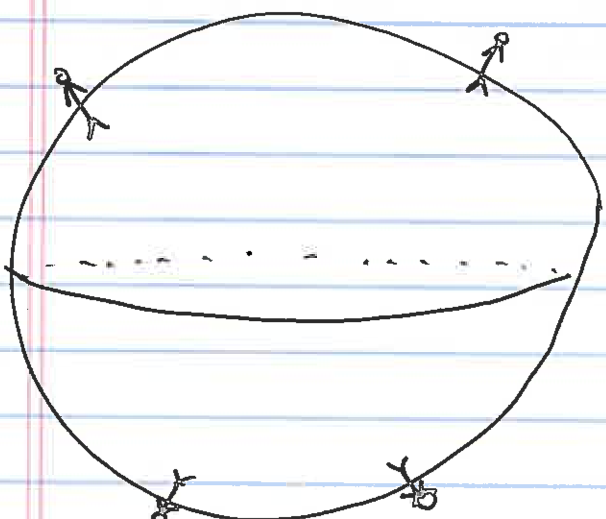
For some surfaces one cannot find a continuous choice of a side for the whole surface.

UNORIENTABLE



Möbius Band has only one side if one walks on it.

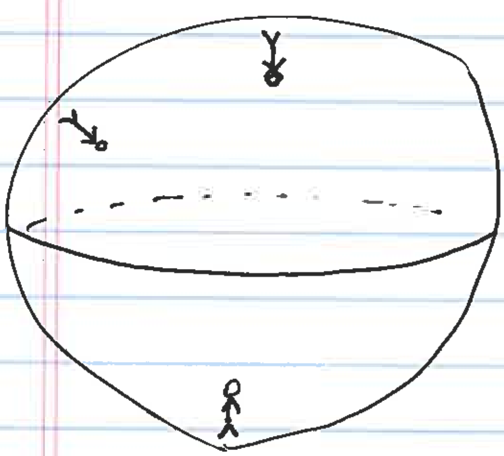
(i)



Sphere has two orientations

(i) Walking on the outside gives outward orientation

ORIENTED



(ii) Walking on the inside gives an inward orientation