IMPORTANT EXAMPLE for 6.2 & 6.3

Ex

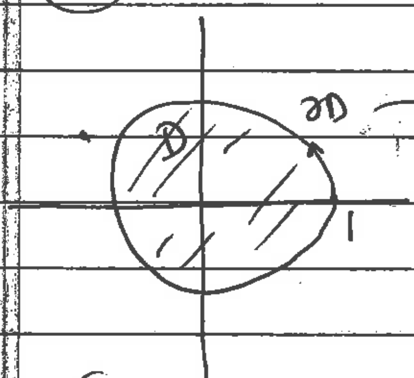
$$F(x,y) = \Big(\underbrace{\frac{-y}{x^2+y^2}}_{P} , \underbrace{\frac{x}{x^2+y^2}}_{Q} \Big) \text{ on } \mathbb{R}^2 - \{(0,0)\}$$

$$\left.\begin{aligned} \frac{\partial Q}{\partial x} &= \frac{1 \cdot (x^2+y^2) - 2x \cdot x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial P}{\partial y} &= \frac{-1 \cdot (x^2+y^2) - 2y \cdot (-y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}. \end{aligned}\right\} \text{ if } (x,y) \neq (0,0)$$

$$\left(\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \implies \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0. \right) \text{ if } (x,y) \neq (0,0)$$

Does Green's Thm apply to F over the unit disc?

No



$\partial D \longrightarrow (\cos t, \sin t) \qquad 0 \le t \le 2\pi$

$x = \cos t \qquad dx = -\sin t \, dt$

$y = \sin t \qquad dy = \cos t \, dt$

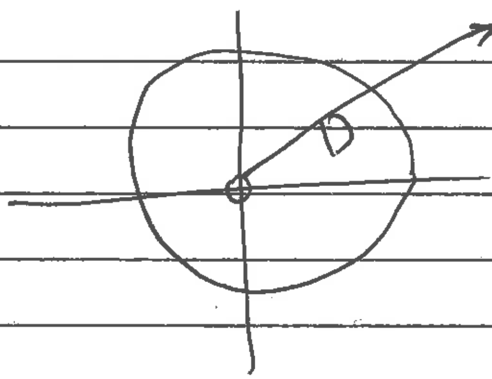
$$\int_{\partial D} P\,dx + Q\,dy = \int_{\partial D} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \int_0^{2\pi} \frac{-\sin t \cdot (-\sin t)\,dt}{1} + \frac{\cos t \cdot \cos t\,dt}{1}$$

$$= \int_0^{2\pi} 1\,dt = 2\pi$$

$$\underbrace{\int P\,dx + Q\,dy}_{2\pi} \neq \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{0} dA$$

1. Green's Thm doesn't apply, since
   P and Q are **not** defined on all of Disc D



P, Q aren't defined at (0,0)

2. Domain P, Q = $\mathbb{R}^2 - \{(0,0)\}$
   not convex

3. Even though $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

$$\int \frac{-y\,dx + x\,dy}{x^2+y^2} = 2\pi$$

$$\Rightarrow \quad F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \text{ is } \textbf{NOT}$$

conservative.

If F were conservative; then $f(\text{end pt}) - f(\text{initial pt}) = 0$

same on a closed curve.

OVERVIEW of Classical Fundamental Theorems:

dimension of ambient space $= n = 1$ $\mathbb{R}^1$ $\qquad D^1 \qquad \partial D = \{a, b\}$.

$k = 1$

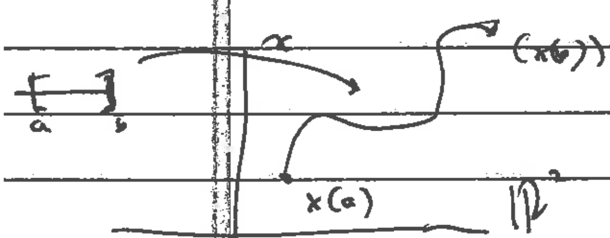
dimension of curve/surface/solid

$\mathbb{R}^1 \quad n = 1$

FTCalculus
$$\underbrace{\int_a^b f'(x)\,dx}_{\text{interior integral}} = \underbrace{f(b) - f(a)}_{\text{boundary}}$$

$n = 2$: $\mathbb{R}^2$

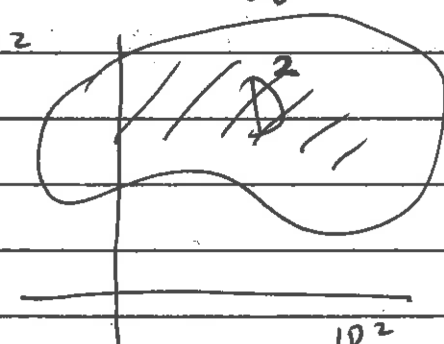
$k = 1$



$n = 2$

$k = 2$

∂D^1



FTLI

$n = 2$
$$\underbrace{\int_{\vec{x}} \nabla f(\vec{x}(t))\,d\vec{x}}_{\text{over the curve}} = \underbrace{f(x(b)) - f(x(a))}_{\text{end pts of the curve}}$$

6.3

Green's Thm:
$$\underbrace{\int_{\partial D} P\,dx + Q\,dy}_{\text{1 dim'l boundary integral}} = \underbrace{\iint_D Q_x - P_y\,d}_{\text{2 dim'l interior integral}}$$

6.2

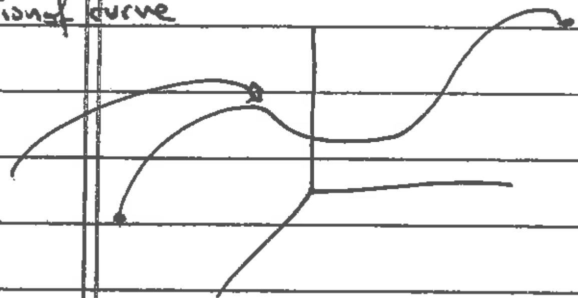
$n = 3 \quad \mathbb{R}^3$

dimension of curve

$k = 1$

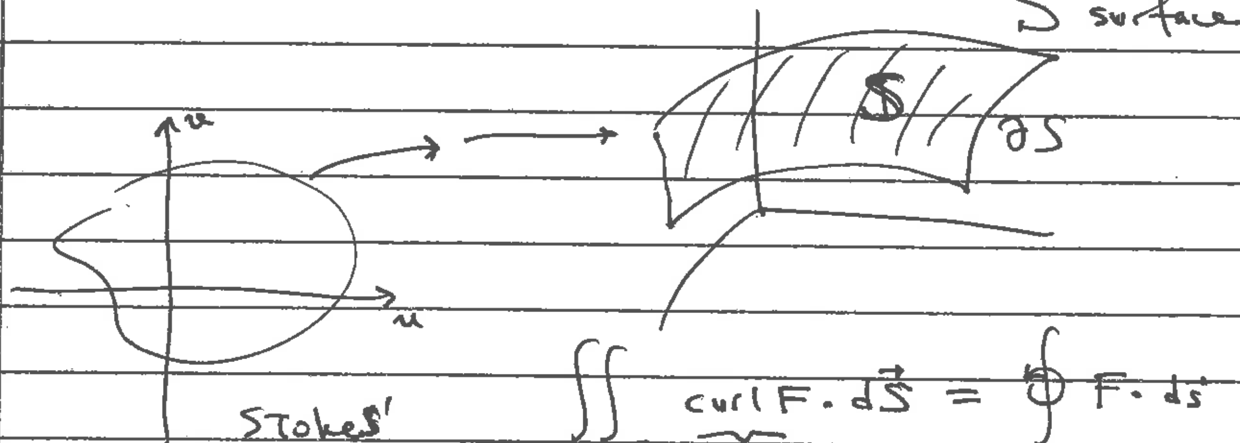
x

$[a,b]$



$$\int_{x(t)} \nabla f \cdot d\vec{s} = f(x(t)) \Big|_{t=a}^{t=b}$$

FTLI (6.3)

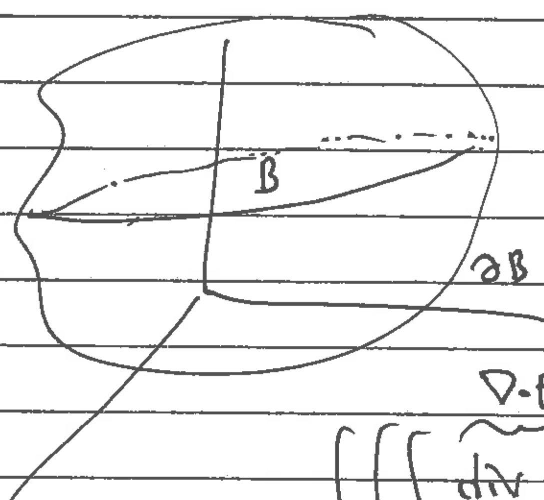


$k = 2$

dim of surface

Stokes' (7.3)

$$\iint_S \underbrace{\text{curl} F}_{\nabla \times F} \cdot d\vec{S} = \oint_{\partial S} F \cdot d\vec{s}$$



$k = 3$

dimension of solid

Gauss' Thm

Divergence Thm

(7.3)

$$\underbrace{\iiint_B \overbrace{\text{div} F}^{\nabla \cdot F}\, dV}_{\text{interior integral}} = \underbrace{\iint_{\partial B} F \cdot d\vec{S}}_{\text{boundary}}$$

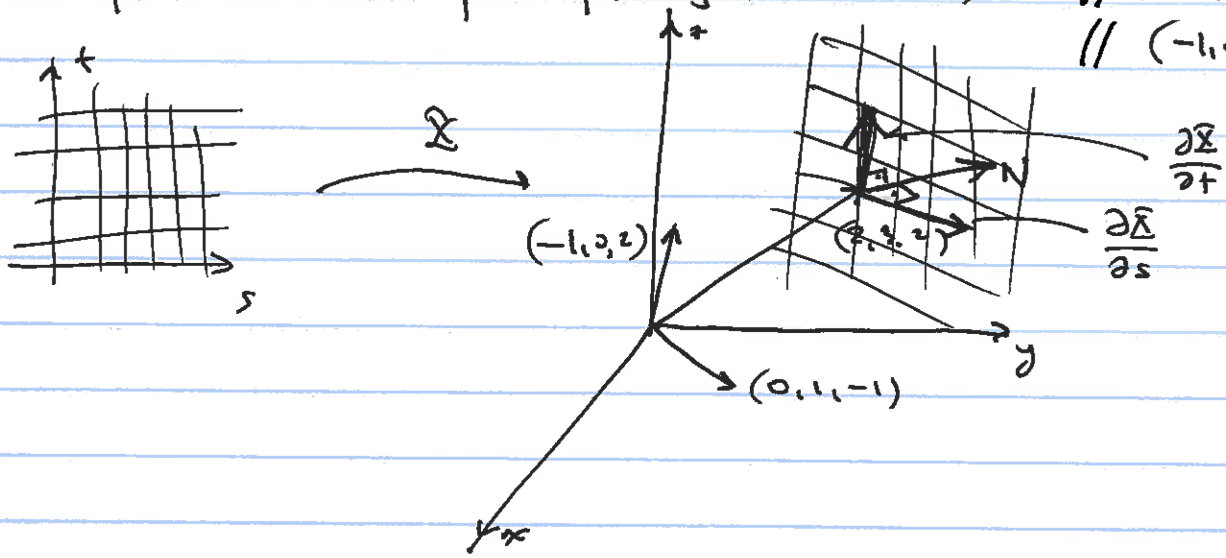
7.1 Parametrized Surfaces

Example

$$(x,y,z) = \mathbf{X}(s,t) = (2,3,2) + s(0,1,-1) + t(-1,0,2)$$

parametrized plane passing thru $(2,3,2)$ ← $\parallel (0,1,-1)$

$\parallel (-1,0,2)$



$$\mathbf{X} : \underset{s,t}{\mathbb{R}^2} \longrightarrow \underset{x,y,z}{\mathbb{R}^3}$$

$$\frac{\partial \mathbf{X}}{\partial s} = (0,1,-1)$$

$$\frac{\partial \mathbf{X}}{\partial t} = (-1,0,2)$$

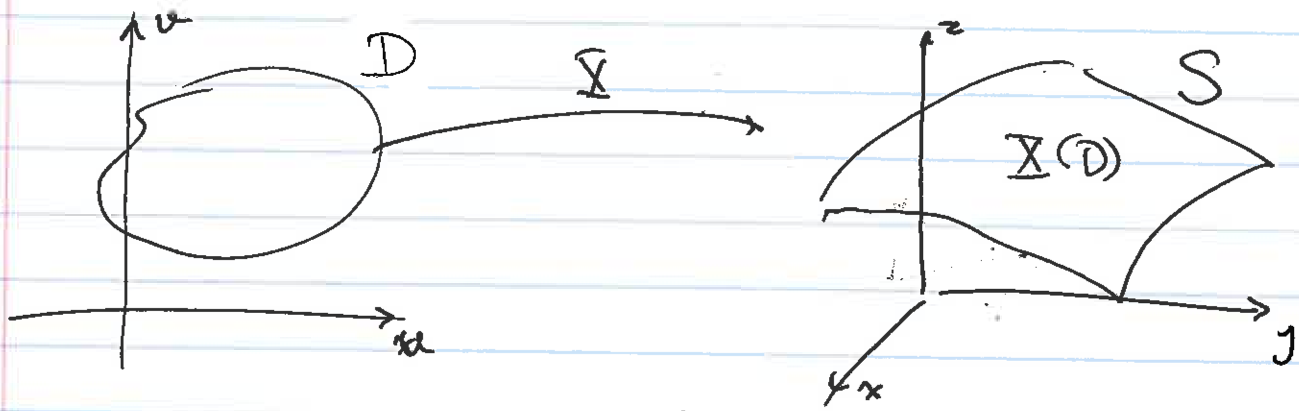
$$N = \frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} = \begin{vmatrix} i & j & k \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{vmatrix}$$

$$= (2,1,1)$$

$$\|N\| = \sqrt{4+1+1} = \sqrt{6}$$

I used (u,v) instead of (s,t) inadvertently. Please take $(s,t)=(u,v)$.

Defn Let $\underset{u,v}{\mathbf{X}: D \overset{\text{open}}{\subseteq} \mathbb{R}^2} \longrightarrow \underset{x,y,z}{\mathbb{R}^3}$ be one-to-one except possibly at the boundary of D



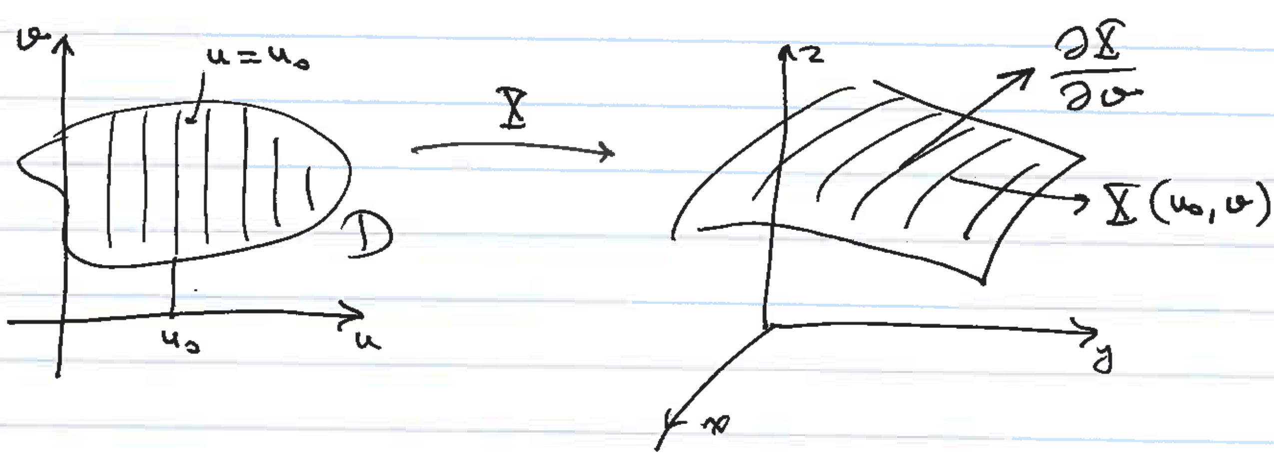
① $\mathbf{X}(D)$ is called a parametrized surface

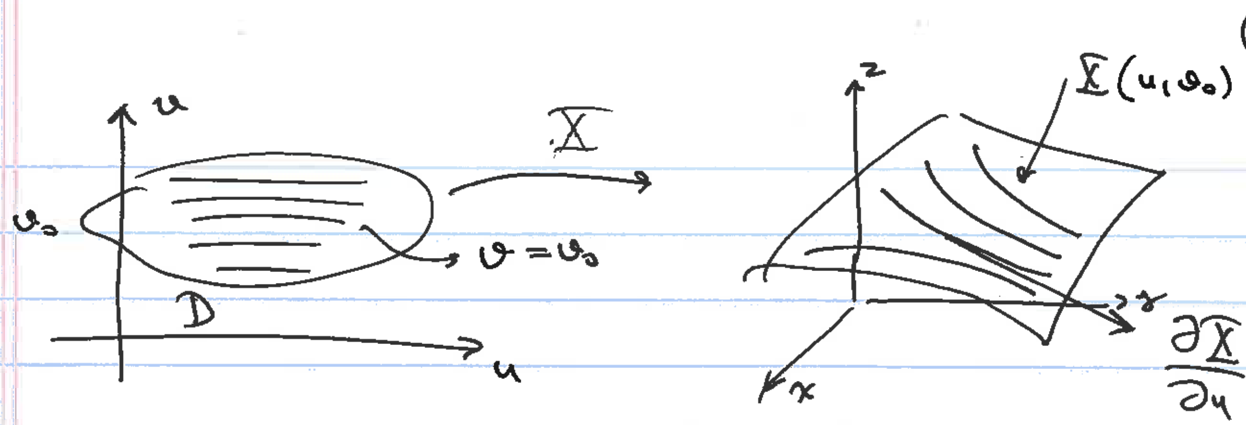
② $\mathbf{X}(D)$ is called a smooth parametrized surface if the following exists and

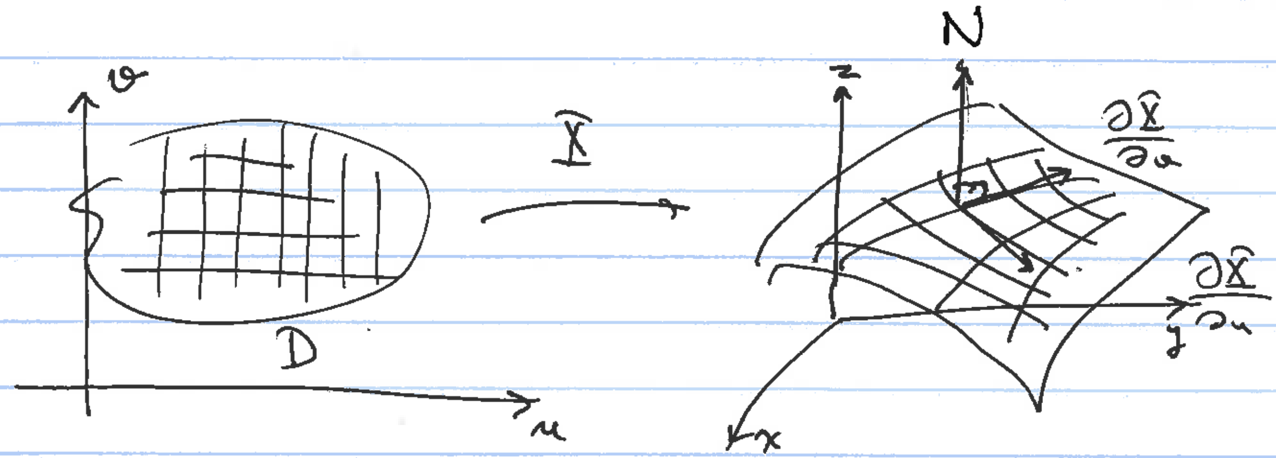
$$\frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} = N(u,v) \neq 0$$

③ $\mathbf{X}(u, v_0)$, $\mathbf{X}(u_0, v)$ are called coordinate curves.

fixed values







$$N = \underbrace{\frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v}}_{\text{at } (u_0, v_0)} \perp \quad S = \mathbf{X}(D) \quad \text{at } \mathbf{X}(u_0, v_0)$$

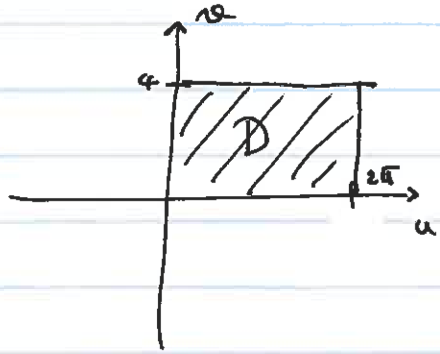
Tangent plane $\perp$ $N(u_0, v_0)$ at $\mathbf{X}(u_0, v_0)$.

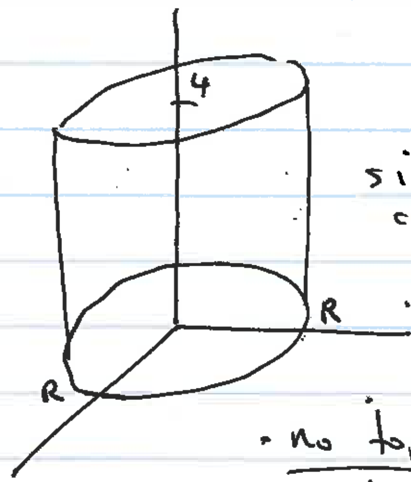
$$\underbrace{N(u_0, v_0)}_{\text{normal}} \cdot \Big[(x, y, z) - \underbrace{\mathbf{X}(u_0, v_0)}_{\text{point of tangency}} \Big] = 0$$

Ex $\underline{X}(u,v) = (\overbrace{R\cos u}^{x}, \overbrace{R\sin u}^{y}, \overbrace{v}^{z})$

R fixed

$$D. \begin{cases} 0 \le u \le 2\pi \\ 0 \le v \le 4 \end{cases}$$





side cylinder

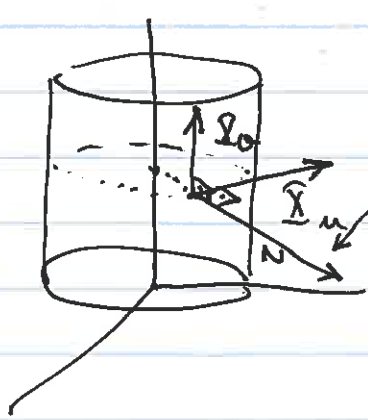
- no top
- no bottom

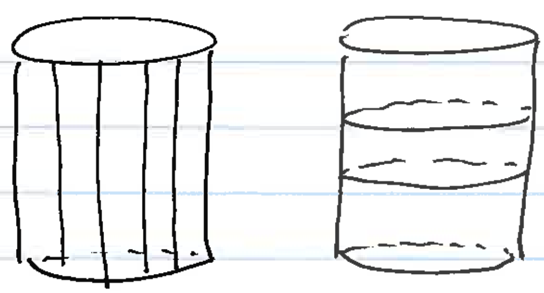
$$\frac{\partial \underline{X}}{\partial u} = (-R\sin u, R\cos u, 0)$$

$$\frac{\partial \underline{X}}{\partial v} = (0, 0, 1)$$

$$N = \frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v} = \begin{vmatrix} i & j & k \\ -R\sin u & R\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$N = (\underbrace{R\cos u, R\sin u}_{\text{horizontal}}, 0)$$





$\underline{X}(u_0, v)$ $\underline{X}(u, v_0)$

Let us take $R = 2$, for the example.

Tangent plane at $\bar{X}\left(\frac{\pi}{4}, 2\right) = \left(2 \cdot \frac{\sqrt{2}}{2}, 2\frac{\sqrt{2}}{2}, 2\right)$

$$= \underbrace{(\sqrt{2}, \sqrt{2}, 2)}_{\text{pt of tangency}}$$

$N = (R\cos u, R\sin u, 0)$

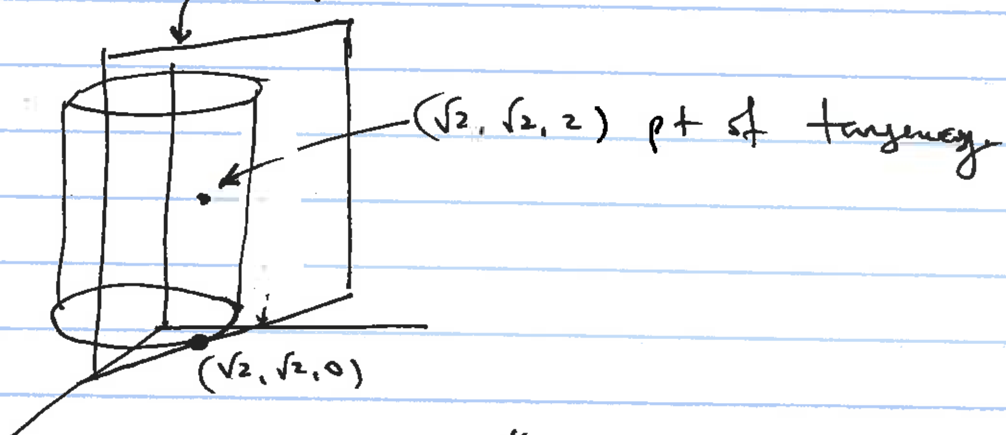
$R=2$, $N\left(\frac{\pi}{4}, 2\right) = (\sqrt{2}, \sqrt{2}, 0)$ normal to S at $(\sqrt{2}, \sqrt{2}, 2)$

Tangent plane:

$$(\sqrt{2}, \sqrt{2}, 0) \cdot \left[(x, y, z) - (\sqrt{2}, \sqrt{2}, 2)\right] = 0.$$

$$\sqrt{2}(x - \sqrt{2}) + \sqrt{2}(y - \sqrt{2}) + 0(z - 2) = 0$$

$x + y = 2\sqrt{2}$.



$$\text{Area} = \iint_D \underbrace{\|N\|\, du\,dv}_{dS} = \int_0^4 \int_0^{2\pi} 2\, du\, dv = 16\pi.$$

$R=2$

$$\|N\| = \sqrt{(R\cos u)^2 + (R\sin u)^2} = R.$$

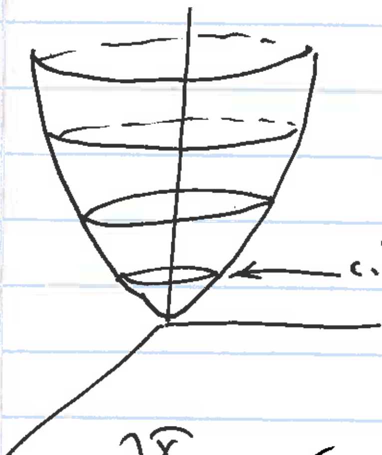
(we took $R = 2$)

7.1 p 468 #7

$$\mathbf{X}(s,t) = (s\cos t,\ s\sin t,\ s^2)$$

$0 \le t \le 2\pi$

$0 \le s$



circle of radius s at height s^2.

$$\begin{array}{cccc} & i & j & k \\ \dfrac{\partial \mathbf{X}}{\partial s} = & (\cos t, & \sin t, & 2s) \end{array}$$

$$\frac{\partial \mathbf{X}}{\partial t} = (-s\sin t,\ s\cos t,\ 0)$$

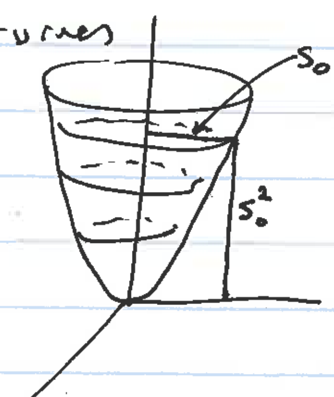
$$N = (-2s^2\cos t,\ -2s^2\sin t,\ s)$$

$$= s(-2s\cos t,\ -2s\sin t,\ 1) \neq 0 \quad \text{we want}$$

<u>S is smooth if $s > 0$.</u>

Coordinate curves

$s = s_0$



$t = t_0$