April 27

(6.3) Conservative vector fields in $\mathbb{R}^3$ ①

(n=3) $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \ \forall_{ij} \iff \text{curl } F = 0.$

Ex) Is $F = (y, 2x - z, z^2)$ conservative?

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & 2x-z & z^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y} z^2 - \frac{\partial}{\partial z}(2x-z), \frac{\partial}{\partial z}(y) - \frac{\partial}{\partial x} z^2, \frac{\partial}{\partial x}(2x-z) - \frac{\partial}{\partial y} y \right)$$

$$= (1, * , *) \neq (0,0,0)$$

(n=3) F conservative $\Longrightarrow$ $\text{curl } F = \vec{0}$

(n=3) F not conservative. $\Longleftarrow$ $\text{curl } F \neq \vec{0}$ (Contrapositive)

Ex $F = (2xy + 3 + z,\ x^2 + ze^{yz} - 4,\ ye^{yz} + x)$

Q: Is F conservative

Q2: If it is, then find a potential function.

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xy+3+z & x^2+ze^{yz}-4 & ye^{yz}+x \end{vmatrix}$$

$$= \Big((e^{yz} + y\cdot ze^{yz} + 0) - (0 + e^{yz} + z\cdot ye^{yz} - 0),$$

$$1 - 1,\ 2x - 2x \Big) = (0,0,0)$$

Domain $= \mathbb{R}^3$, which is convex

$\Rightarrow$ F is conservative.

$\Rightarrow \exists f \quad \nabla f = (f_x, f_y, f_z) = F$

$f_x = 2xy + 3 + z \longrightarrow f = \int (2xy + 3 + z)\,dx = x^2y + 3x + xz + c_1(y,z)$

$f_y = x^2 + ze^{yz} - 4 \longrightarrow f = \int (x^2 + ze^{yz} - 4)\,dy = x^2y + e^{yz} - 4y + c_2(x,z)$

$f_z = ye^{yz} + x \longrightarrow f = \int (ye^{yz} + x)\,dz = e^{yz} + xz + c_3(x,y)$

$f = x^2y + 3x + xz + e^{yz} - 4y +$ any real #

6.3 p449 #28

Find $\displaystyle\int_C (2y-3z)\,dx + (2x+z)\,dy + (y-3x)\,dz$.

C: $(0,0,0) \to (0,1,1)$ line segment, followed by $(0,1,1) \to (1,2,3)$ line segment

Want to apply FTLI. Is F conservative?

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2y-3z & 2x+z & y-3x \end{vmatrix} = (1-1, -3+3, 2-2) = (0,0,0).$$

Domain $F = \mathbb{R}^3$ convex

together

F is conservative.

Now: Want f s.t. $\nabla f = F$

$f_x = 2y - 3z \qquad f = 2xy - 3xz + c_1(y,z)$

$f_y = 2x + z \qquad f = 2xy + yz + c_2(x,z)$

$f_z = y - 3x \qquad f = yz - 3xz + c_3(x,y)$

$$f = 2xy - 3xz + yz + 5$$

Doesn't matter what it is, as long as a real #.

By FTLI

$$\int (2y-3z)\,dx + (2x+z)\,dy + (y-3x)\,dz$$

$$= \int_C \vec{\nabla} f \cdot \vec{ds} = f \Big|_{\text{initial pt}}^{\text{end pt}}$$

$$= 2xy - 3xz + yz + 5 \Big|_{(0,0,0)}^{(1,2,3)}$$

$$= (4 - 9 + 6 + \cancel{5}) - (0 + 0 + 0 + \cancel{5})$$

$$= 1$$

arbitrary constant cancels out.