April 26

(6.2) ①

<u>How to calculate area of a region by using Green's Thm.</u>

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \Big(\underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\text{want}}\Big)\,dA$$

$$= \iint_D 1 \cdot dA = \text{Area } D$$

Choose P, Q appropriately

<u>Choose</u> $\left.\begin{matrix} Q = x \\ P = 0 \end{matrix}\right\} \Rightarrow \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1 \Leftarrow \left\{\begin{matrix} Q = 0 \\ P = -y \end{matrix}\right.$ <u>Choose</u>

$$\oint_{\partial D} x\,dy = \text{Area of } D. = \oint_{\partial D} -y\,dx$$

$$= \oint \tfrac{1}{2}(x\,dy - y\,dx)$$

All are correct

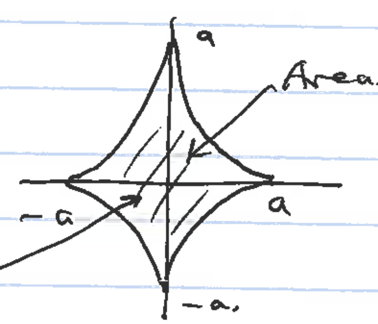
6.2 <u>Exc #14</u> p 437

$x = a\cos^3 t \qquad 0 \le t \le 2\pi$

$y = a\sin^3 t$

$$x^{2/3} + y^{2/3} = 1.\ a^{2/3}$$

$$y = (a^{2/3} - x^{2/3})^{3/2}$$



Find Area

$$\text{Area} = \oint_{\partial D} \frac{1}{2}\left(x\,dy - y\,dx\right)$$

$$x = a\cos^3 t \qquad dx = -3a\cos^2 t \sin t$$

$$y = a\sin^3 t \qquad dy = 3a \sin^2 t \cos t$$

$$\frac{1}{2}\left(x\,dy - y\,dx\right) = \frac{1}{2}\left(a\cos^3 t \cdot 3a \sin^2 t \cos t + a\sin^3 t \, 3a\cos^2 t \sin t\right)$$

$$= \frac{3a^2}{2}\left(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t\right)$$

$$= \frac{3a^2}{2} \cos^2 t \sin^2 t \,\underbrace{\left(\cos^2 t + \sin^2 t\right)}_{1}$$

$$\sin 2t = 2\sin t \cos t$$

$$= \frac{3a^2}{2}\left(\frac{1}{2}\sin 2t\right)^2$$

$$= \frac{3a^2}{8}\sin^2 2t = \frac{3a^2}{8}\,\frac{1-\cos 4t}{2}$$

$$\text{Area} = \int_0^{2\pi} \frac{3a^2}{16}\left(1-\cos 4t\right) dt$$

$$= \frac{3a^2}{16}\cdot 2\pi - \cancelto{0}{\frac{3a^2}{16}\int_0^{2\pi} \cos 4t \, dt}$$

$$= \frac{3a^2\pi}{8}$$

6.3 Defn A vector field $F : U^{\text{open}} \subseteq \mathbb{R}^n \to \mathbb{R}^n$

is called conservative (or gradient field)

if $\exists f : U^{\text{open}} \subseteq \mathbb{R}^n \to \mathbb{R}$ s.t

$$\nabla f = F$$

We call F conservative / gradient field
f a potential function

Ex ① $\nabla(\underbrace{x^3y}_{\text{potential}}) = \underbrace{(3x^2y, x^3)}_{\text{conservative}}$ if $x^3y : \mathbb{R}^2 \to \mathbb{R}$.

② $F = (x, x)$ is not conservative

Suppose $\exists f$ $\nabla f = F = (x, x)$

$\nabla f = (f_x, f_y)$

$f_x = x \longrightarrow f_{xy} = 0$

$f_y = x \qquad\quad f_{yx} = 1$

$f_{xy} \neq f_{yx}$ Contradiction.

Chap 2 Recall f is twice continuously diff'ble $\Rightarrow f_{xy} = f_{yx}$

Ex 3 $f = \dfrac{1}{\|x\|} = (x^2+y^2+z^2)^{-\frac{1}{2}} : \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R}$.

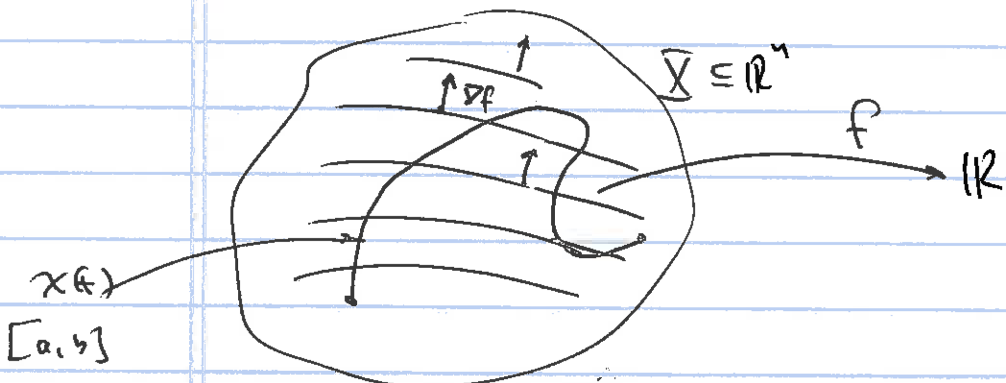
$$\nabla f = \Big(2x\,(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot \left(-\tfrac{1}{2}\right),\; 2y\,(x^2+y^2+z^2)^{-\frac{3}{2}} \left(-\tfrac{1}{2}\right),\; 2z\,(x^2+y^2+z^2)^{-\frac{3}{2}} \left(-\tfrac{1}{2}\right) \Big)$$

$$= (-x, -y, -z) \cdot (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= -\frac{\vec{x}}{\|x\|^3} \quad \left(\text{in magnitude} : \frac{1}{R^2} \right)$$

$R = \sqrt{x^2+y^2+z^2}$

(*) (FTLI) Fundamental Thm of Line integrals:



Let $f : X^{\text{open}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuously diff'ble

Let $\vec{x} : [a,b] \longrightarrow X$ be a piecewise diff'ble curve

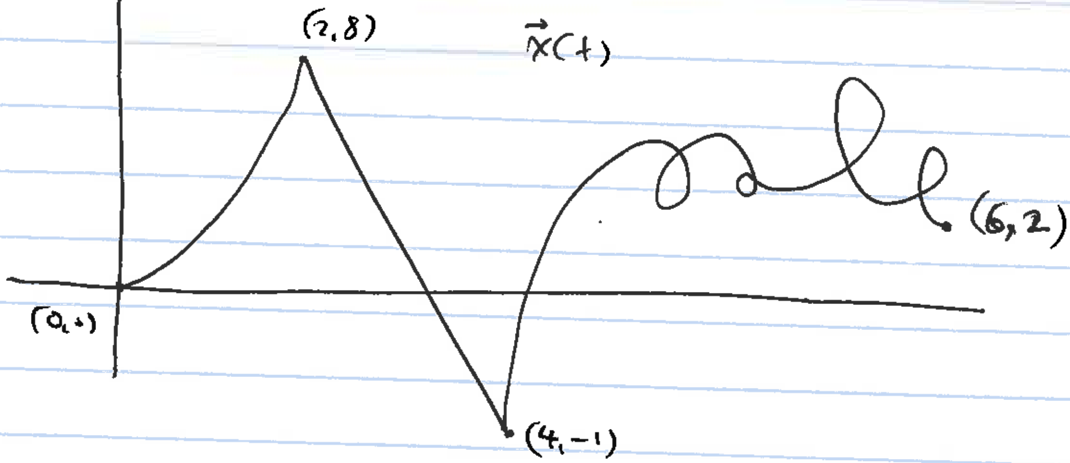
Then

$$\int_{\vec{x}} \vec{\nabla} f \cdot \vec{ds} = f(\vec{x}(b)) - f(\vec{x}(a))$$

Find

Ex $$\int_{\vec{x}(t)} 3x^2y\,dx + x^3\,dy = \text{I}$$

where $\vec{x}(t)$ is:



recall $\nabla(x^3y) = (3x^2y,\ x^3)$

$$\text{I} = x^3y\Big|_{(0,0)}^{(6,2)} = 6^3\cdot 2 - 0^3\cdot 0 = 432$$

CRUCIAL:

Q: 1 How do I check a v.f. is conservative?

2 How do I find a potential function?

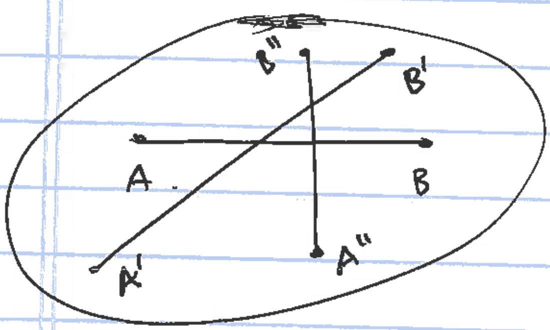
Thm: Let $F(\overbrace{x_1,\dots x_n}^{n \text{ variables}}) = (\underbrace{F_1,\dots F_n}_{n \text{ components}})$

Let $F: U^{\text{open}} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be cont. diff'ble.

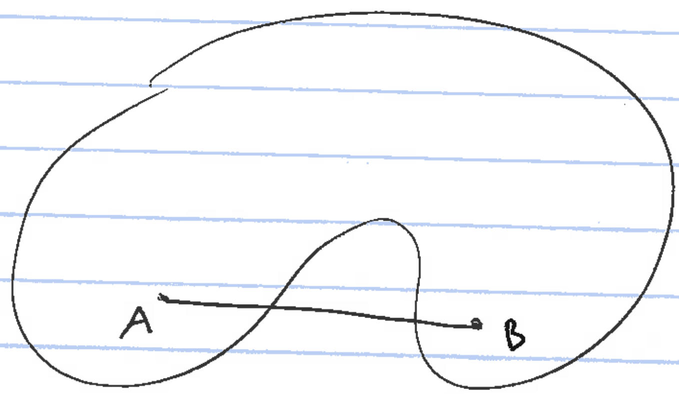
1. F conservative $\Longrightarrow$ $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i} \quad \forall\, i,j$

2. F conservative $\Longleftarrow$ $\begin{cases} \dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i} \quad \forall\, i,j \\ U \text{ convex region} \end{cases}$

Def U is convex if $\underset{\text{for all}}{\forall} A, B \in U$, $\underbrace{\overline{AB}}_{\text{line segment}} \subseteq U$



convex



not convex

$n=2$

$$F = (P, Q) = F(x,y)$$
$$P = F_1,\quad Q = F_2,\quad x = x_1,\quad y = x_2$$

$$\forall i,j \quad \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \iff \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Exc #4

$$F = \underbrace{2x\sin y}\, i + \underbrace{x^2\cos y}\, j$$

(1) Is F conservative?

(2) Find potential function, if yes for (1).

Check: (a) $\frac{\partial}{\partial y}(2x\sin y) \overset{?}{=} \frac{\partial}{\partial x}(x^2\cos y)$

$$2x\cos y = 2x\cos y$$

Check (b) Domain $U = \mathbb{R}^2$ convex.

$\underline{\text{Thm}} \implies F$ is conservative.

(2) Find potential function f

$$F = (2x\sin y,\ x^2\cos y) = \nabla f = (f_x, f_y)$$

$$f_x = 2x\sin y \longrightarrow f = \int f_x\, dx = \int 2x\sin y\, dx = x^2\sin y + C_1(y)$$

$$f_y = x^2\cos y \longrightarrow f = \int f_y\, dy = \int x^2\cos y\, dy = x^2\sin y + C_2(x)$$

Can take $f = x^2\sin y + C \quad (C\in\mathbb{R})$

6.1 Exc # 24 p 226(b)

Similar ~ $F = (x, y, z)$

$f = \frac{1}{2}(x^2+y^2+z^2)$ $\nabla f = F$

$\frac{1}{2}(x^2+y^2+z^2) = c$

spheres of radius $\sqrt{2c}$