April 24, 2017

(1)

FUNDAMENTAL THEOREMS

- Calc I (FTC) $\int_a^b f'(x)\,dx = f(b) - f(a)$

6.2 • GREEN'S THM

6.3 • FT Line Integrals

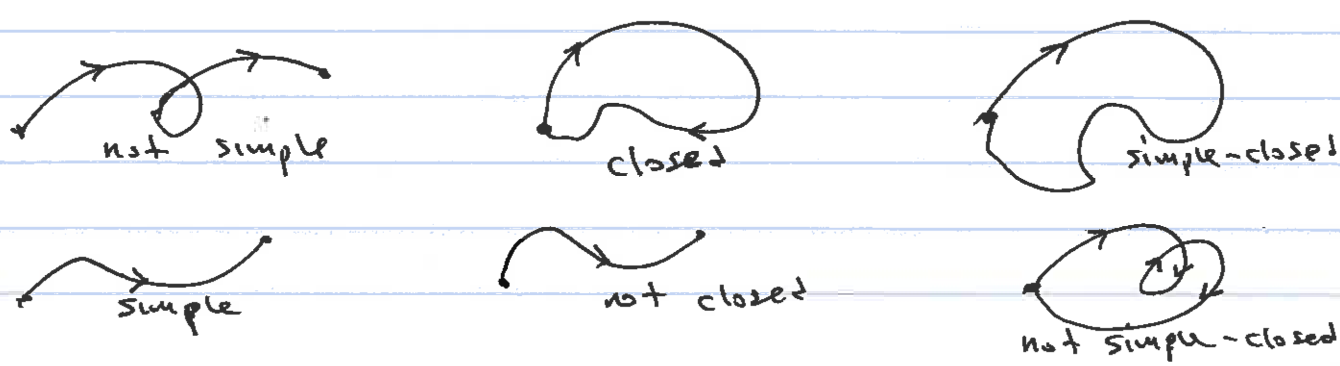
7.3 • STOKES' THM

7.3 • GAUSS' THM

(6.2) **Defn** A curve $\vec{x} : [a,b] \to \mathbb{R}^n$ is called

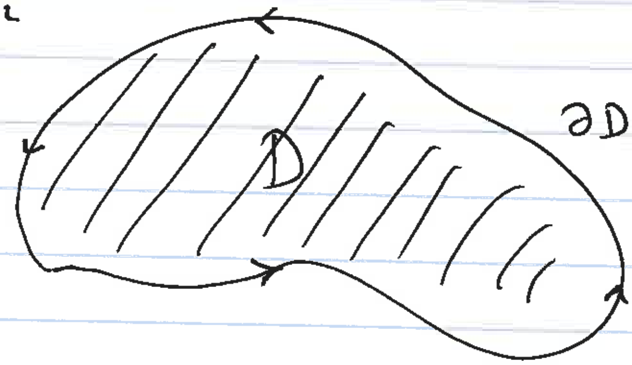
- simple if $\vec{x}$ is 1-1;
- closed if $\vec{x}(a) = \vec{x}(b)$;
- simple-closed if (i) x is closed, and
  (ii) $\vec{x}|[a,b)$ is 1-1.

Ex.



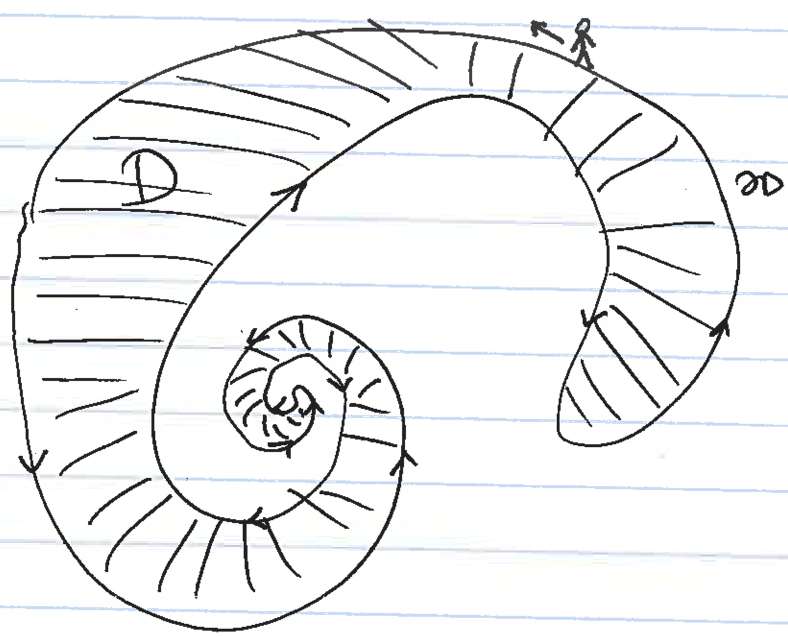
How to positively orient boundary?

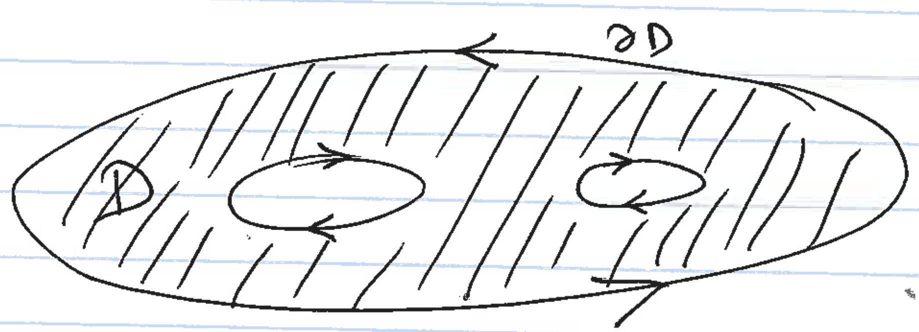
$D \subseteq \mathbb{R}^2$



∂D = boundary of D

positively oriented [If one walks along the boundary in the positive direction then D stays to the left





THEOREM (GREEN'S)

Let D be a closed and bounded region in $\mathbb{R}^2$, whose boundary ∂D consists of finitely many simple-closed curves which are piecewise diff'ble, and positively oriented.

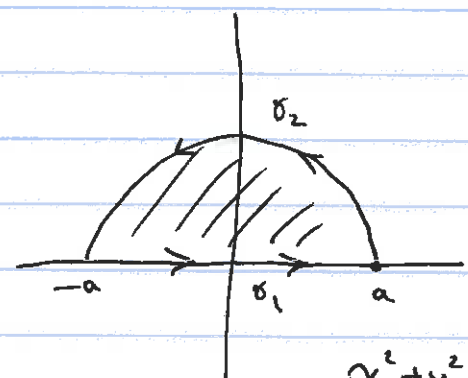
Let $P, Q : D \to \mathbb{R}$ be defined on all of D and be continuously diff'ble.

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

---

p 436 Exc # 4.

$$F(x,y) = 2y\vec{i} + x\vec{j}$$
$$F = (2y, x) = (P, Q)$$
$$P = 2y$$
$$Q = x$$



$x^2+y^2 \le a^2$
$y \ge 0$

$$\partial D: \begin{cases} \gamma_1(t) = (t, 0) & -a \le t \le a \\ \gamma_2(t) = (a\cos t, a\sin t) & 0 \le t \le \pi \end{cases}$$

First γ_1:

$$\int_{\gamma_1} P\,dx + Q\,dy = \int_{-a}^{a} \overset{2y}{0} \cdot \overset{dx}{dt} + \overset{x}{t} \cdot \overset{dy}{0} = 0$$

$$\gamma_1 \begin{cases} x = t & dx = dt \\ y = 0 & dy = 0 \end{cases}$$

$P = 2y$

$Q = x$

$\gamma_2 : \begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$ $\quad dx = -a\sin t\, dt$, $\quad dy = a\cos t\, dt$

Next γ_2:

$$\int_{\gamma_2} P\,dx + Q\,dy = \int_0^{\pi} \underbrace{(2a\sin t)}_{\substack{2y \\ \parallel \\ P}} \underbrace{(-a\sin t)}_{dx} + \underbrace{(a\cos t)}_{\substack{x \\ \parallel \\ Q}} \cdot \underbrace{(a\cos t)\,dt}_{dy}$$

$$= \int_0^{\pi} a^2 \left[-2\sin^2 t + \cos^2 t \right] dt$$

$$= \int_0^{\pi} a^2 \left[-2 \cdot \frac{1-\cos 2t}{2} + \frac{1+\cos 2t}{2} \right] dt$$

$$= a^2 \int_0^{\pi} \left(-1 + \cos 2t + \frac{1}{2} + \frac{1}{2}\cos 2t \right) dt$$

$$= a^2 \int_0^{\pi} -\frac{1}{2} + \frac{3}{2}\cos 2t \; dt = a^2 \left(-\frac{\pi}{2} \right) + \underbrace{\left(\frac{3}{4}\sin 2t \Big|_0^{\pi} \right)}_{0}$$

$$= -\frac{\pi a^2}{2}.$$

$$\oint_{\partial D} P\,dx + Q\,dy = \left(\int_{\gamma_1} + \int_{\gamma_2} \right)(P\,dx + Q\,dy) = 0 - \frac{\pi a^2}{2}$$

Next

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

They match ✓ verified ✓

$P = 2y$

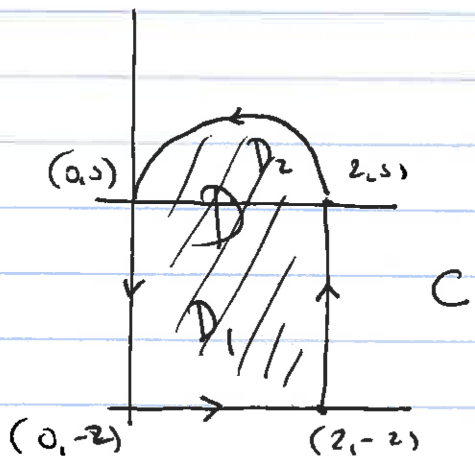
$Q = x$

$$= \iint_D (-1)\, dA = -\text{ area } D$$

$$= -\frac{\pi a^2}{2}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 2 = -1$$

Exc # 8 p 437.



$\int_C \vec{F} \cdot d\vec{s}$ want.

$F = (\underbrace{3xy}_{P}, \underbrace{2x^2}_{Q})$.

$$\int_{\partial D = C} F \cdot ds = \pm \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

check Boundary orientation, positive ↺ yes!

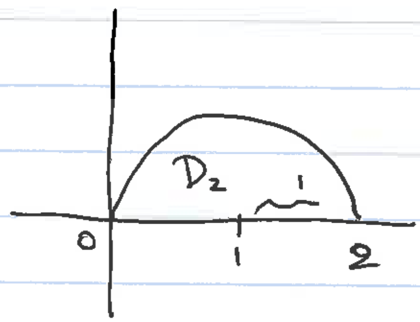
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4x - 3x = x.$$

$$\iint_D x \, dA = \left(\iint_{D_1} + \iint_{D_2} \right) x \, dA$$

$$\iint_{D_1} x \, dA = \int_0^2 \int_{-2}^0 x \; dy \, dx = \int_0^2 xy \Big|_{-2}^{0} dx$$

$$= \int_0^2 2x \, dx = x^2 \Big|_0^2 = 4.$$

$$\iint_{D_2} x\, dA = \int_0^2 \int_0^{\sqrt{1-(x-1)^2}} x \, dy\, dx$$



$(x-1)^2 + y^2 = 1.$

$$= \int_0^2 x\sqrt{1-(x-1)^2}\, dx. = \int_{-1}^{1} (u+1)\sqrt{1-u^2}\, du.$$

$x - 1 = u.$

$dx = du$

$$= \int_{-1}^{1} \underbrace{u\sqrt{1-u^2}}_{\text{odd}}\, du + \int_{-1}^{1} 1\sqrt{1-u^2}\, du$$

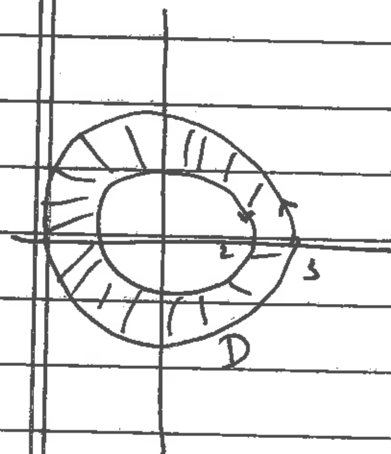
$$= 0 + \frac{1}{2}\left(\underbrace{\text{area of disc of radius } 1}_{\pi}\right)$$

$$= \frac{\pi}{2}$$

$$\int_C P\,dx + Q\,dy = \iint_{D = D_1 \cup D_2} x\, dA = 4 + \frac{\pi}{2}.$$

Exc #6 p 437.

$$\int F \cdot ds = \int (\underbrace{x^2y + x}_{P},\ \underbrace{y^3 - xy^2}_{Q}) \cdot \vec{x}'(t)\, dt$$



$$\frac{\partial Q}{\partial x} = -y^2$$

$$\frac{\partial P}{\partial y} = x^2.$$

Green's Thm $\displaystyle\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$$= \iint_D (-y^2 - x^2)\, dA$$

Use polar Coordinates

Jacobian

$$= \int_0^{2\pi}\int_2^3 -r^2 \cdot r \cdot dr\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_2^3 -r^3\, dr\right)$$

$$= 2\pi \cdot \left.\frac{-r^4}{4}\right|_2^3 = -\frac{\pi}{2}(81 - 16) = \frac{-65\pi}{2}.$$