April 21

6.1 $\quad \vec{x} : [a,b] \longrightarrow \mathbb{R}^n$ $\quad$ 1

Recall $$\int_{x} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}\,'(t)\,dt = \int_{\vec{x}} \vec{F} \cdot \underbrace{\vec{T}\,ds}_{d\vec{s}}$$

Another way to represent same integral

$$\int_{\vec{x}} P\,dx + Q\,dy + R\,dz = \int_{\vec{x}} \underbrace{(P,Q,R)}_{\vec{F}} \cdot \underbrace{(dx, dy, dz)}_{\vec{x}\,'(t)\,dt}$$

$$= \int_{\vec{x}} \underbrace{(P,Q,R)}_{F} \cdot \underbrace{\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)}_{\vec{x}\,'(t)} dt$$

Ex $$\int_{\vec{x}} y^2\,dx + xz\,dy - z^2\,dz$$

$$\vec{x}(t) = (\underbrace{2t^2}_{x}, \underbrace{t-1}_{y}, \underbrace{3t}_{z}) \qquad 0 \le t \le 2$$

$$\left.\begin{aligned} x &= 2t^2 \\ y &= t-1 \\ z &= 3t \end{aligned}\right\} \longrightarrow \quad \begin{aligned} dx &= 4t\,dt \\ dy &= dt \\ dz &= 3\,dt \end{aligned}$$

$$\int_0^2 (t-1)^2\, 4t\,dt + \underbrace{(2t^2)(3t)}\,dt - (3t)^2 \cdot 3\,dt$$

$$= \int_0^2 \underbrace{\left((t-1)^2, (2t^2)(3t), -3t^2\right)}_{F(\vec{x}(t))} \cdot \underbrace{(4t, 1, +3)}_{\vec{x}\,'(t)}\,dt$$

$$= \int_0^2 \left((t^2 - 2t + 1)\,4t + 6t^3 - 27t^2 \right) dt$$

$$= \int_0^2 \left(4t^3 - 8t^2 + 4t + 6t^3 - 27t^2 \right) dt$$

$$= \int_0^2 \left(10t^3 - 35t^2 + 4t \right) dt$$

$$= \frac{10t^4}{4} - \frac{35}{3}t^3 + \frac{4t^2}{2} \Big|_0^2$$

$$= 40 - \frac{280}{3} + 8 = 48 - \frac{280}{3}$$

$$= \frac{144 - 280}{3} = -\frac{136}{3}.$$

What is F?

$$F = (y^2, xz, -z^2)$$

$$F(\vec{x}(t)) = F(2t^2, t-1, 3t)$$

$$= \left((t-1)^2, (2t^2)(3t), -(3t)^2 \right)$$

Reparametrizations:

$(\text{first})^2 = \text{second}$

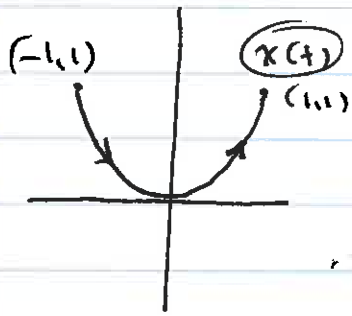
$x^2 = y$

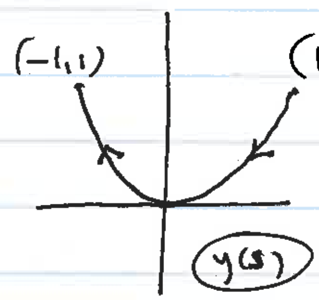
Ⓘ $\vec{x}(t) = (t, t^2) \qquad -1 \le t \le 1$

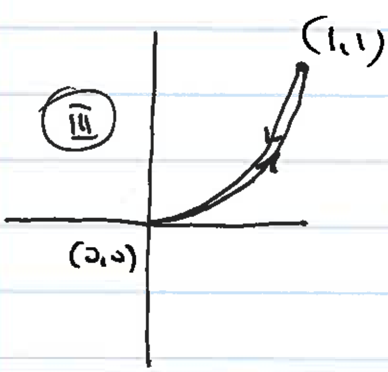
Ⓘⓘ $\vec{y}(s) = (1-s, (s-1)^2) \qquad 0 \le s \le 2 \qquad t = 1-s$

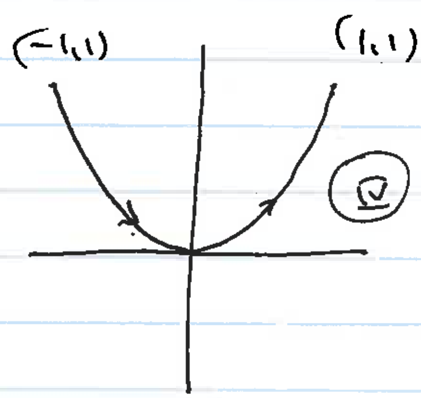
Ⓘⓘⓘ $\vec{\alpha}(u) = (u^2, u^4) \qquad -1 \le u \le 1 \qquad t = u^2$

Ⓘⓥ $\beta(w) = (w^3, w^6) \qquad -1 \le w \le 1 \qquad t = w^3$









ⓐ $\int f\, ds \qquad f$ real valued $\qquad$ I, II, IV will be same answer

ⓑ $\int \vec{F}\cdot d\vec{s} \qquad \int_{I} \vec{F}\cdot d\vec{s} = \int\int_{IV} \vec{F}\cdot d\vec{s} = -\int\int_{II} \vec{F}\cdot d\vec{s}$

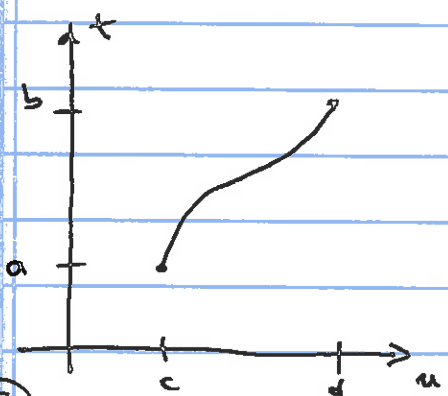
In General $\Big\{$ III You'll get different answers in ⓐ than I, II, IV

III " " " " in ⓑ than I, II, IV

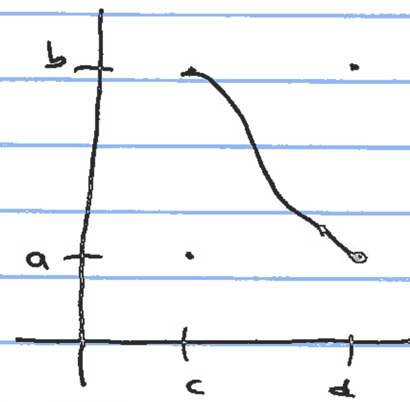
Def^n Let $\vec{x}(t) : [a,b] \to \mathbb{R}^n$ be a diff'ble parametrized curve.

We want to make $t = f(u)$ substitution.

$\underset{u}{f : [c,d]} \to \underset{t}{[a,b]}$, be 1-1, onto



$t = f(u)$



I $\qquad$ II

$f' > 0 \qquad f' < 0$

$f' > 0 \qquad f' < 0$

$\boxed{\begin{array}{l} \text{Change variables : } \vec{y}(u) = \vec{x}(f(u)) = \vec{x}(t) \\ \vec{y} \text{ is a reparametrization of } \vec{x} \end{array}}$

I $\vec{x}$ and $\vec{y}$ go in the same direction | $\vec{x}$ & $\vec{y}$ go in opposite directions. II

Prop Let $\vec{x}(t)$ and $\vec{y}(u)$ be reparametrizations of each other as above, both in $X \subseteq \mathbb{R}^n$.

(i) For $g : X \subseteq \mathbb{R}^n \to \mathbb{R}$: $\displaystyle\int_{\vec{x}} g \cdot ds = \int_{\vec{y}} g \cdot ds$ for scalar line integrals

(ii) For $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \pm \int_{\vec{y}} \vec{F} \cdot d\vec{s}$$

where:
+ if I as above $f' > 0$
− if II as above $f' < 0$.

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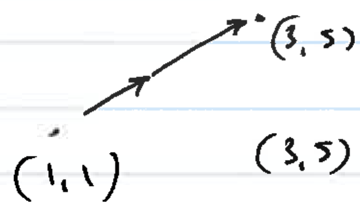
$$\int (x^2-y)\,dx + (x-y^2)\,dy$$

C line segment from $(1,1)$ to $(3,5)$



$(3,5)-(1,1) = (2,4)$

$\vec{x}(t) = (1,1) + t(2,4) \qquad 0 \le t \le 1$

$= (1+2t,\ 1+4t)$

$x = 1+2t \qquad dx = 2dt$

$y = 1+4t \qquad dy = 4dt$

$$\int_0^1 \left((1+2t)^2 - (1+4t)\right)\cdot 2dt + \left((1+2t) - (1+4t)^2\right)\cdot 4dt$$

$$= \int_0^1 (\cancel{1} + \cancel{4t} + 4t^2 - \cancel{1} - \cancel{4t})\, 2dt + (\cancel{1} + 2t - \cancel{1} - 8t - 16t^2)\, 4dt$$

$$= \int_0^1 (8t^2 - 24t - 64t^2)\, dt$$

$$= \int_0^1 (-56t^2 - 24t)\, dt = -\frac{56}{3}t^3 - \frac{24}{2}t^2 \Big|_0^1$$

$$= \frac{-56}{3} - 12 = \frac{-92}{3}.$$

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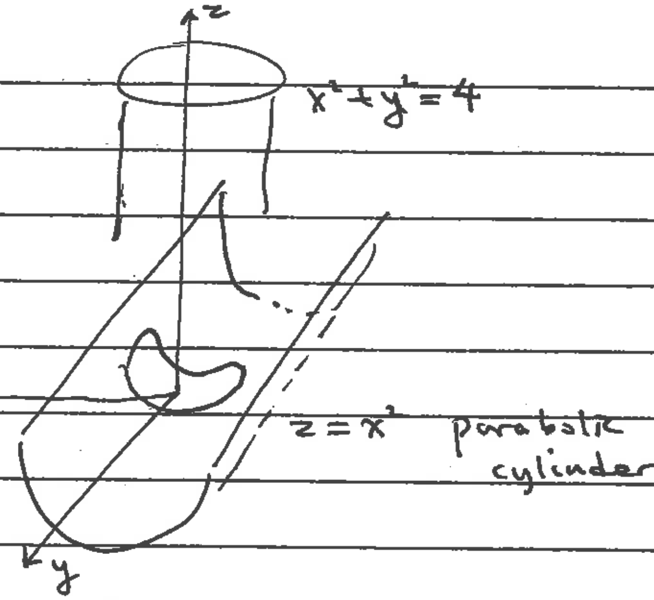
curve of intersection $\begin{cases} z = x^2 \\ x^2+y^2 = 4 \end{cases}$



parametrize: $(\underbrace{\overbrace{2\cos t}^{x},\ \overbrace{2\sin t}^{y}}_{\substack{x^2+y^2=4 \\ \text{top view}}},\ \underbrace{4\cos^2 t}_{z=x^2})$ $0 \le t \le 2\pi$

Calculate: $\int_C z\,dx + x\,dy + y\,dz$

$$= \int_0^{2\pi} \underbrace{(4\cos^2 t)}_{z}\underbrace{(-2\sin t\,dt)}_{dx} + \underbrace{(2\cos t)}_{x}\underbrace{(2\cos t)dt}_{dy} + \overbrace{2\sin t}^{y} \cdot \underbrace{(-8\cos t\sin t}_{dz} \ldots$$

$$= \int_0^{2\pi} 4\cos^2 t\,dt = \int_0^{2\pi} 2(1+\cos 2t)\,dt = 4\pi, \text{ since}$$

$$\int_0^{2\pi} \cos^2 t \sin t\,dt = 0 = \int_0^{2\pi} \cos t \sin^2 t\,dt$$