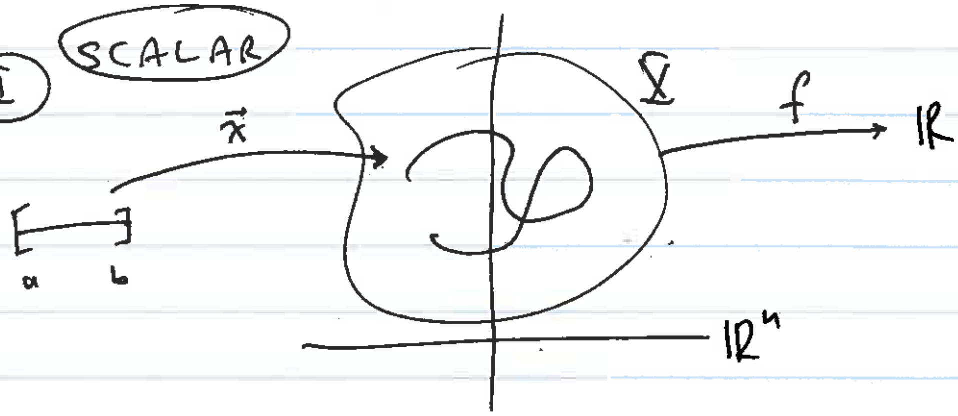
6.1

There are two types of line integrals

Ⅰ Scalar line integrals
Ⅱ Vector line integrals

Ⅰ SCALAR




One defines scalar line integral of f: $\int_{\vec{x}} f\,ds = \int_a^b f(\vec{x}(t)) \underbrace{\|\vec{x}'(t)\|\,dt}_{ds}$

Ex. 0 $\int_{\vec{x}} 1 \cdot ds$ = length of $x(t)$.

p 426 Exc #2

$f(x,y,z) = xyz$

$\vec{x}(t) = (t, 2t, 3t)$ $\quad 0 \le t \le 2$

Line segment from $(0,0,0)$ to $(2,4,6)$

$\vec{x}'(t) = (1, 2, 3)$

$\|x'(t)\| = \sqrt{1+4+9} = \sqrt{14}.$

$$\int f\,ds = \int_0^2 xyz\,ds = \int_0^2 t \cdot 2t \cdot 3t \underbrace{\sqrt{14}\,dt}_{ds}$$

$$\vec{x}(t) = (t, 2t, 3t)$$
$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ x & y & z \end{array}$$

$$= \int_0^2 6\sqrt{14}\,t^3\,dt = \frac{6\sqrt{14}\,t^4}{4}\Big|_0^2 = 24\sqrt{14}.$$

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Exc #4 p 426 $\quad f(x,y,z) = 3x + xy + z^3$

$$\vec{x}(t) = (\underbrace{\cos 4t}_{x}, \underbrace{\sin 4t}_{y}, \underbrace{3t}_{z}) \quad 0 \le t \le 2\pi$$

$$\int_{\vec{x}} f\,ds = ?$$

$$\vec{x}'(t) = (-4\sin 4t, 4\cos 4t, 3)$$

$$|\vec{x}'| = \sqrt{16\sin^2 4t + 16\cos^2 4t + 9} = 5$$

$$ds = 5\,dt$$

$$f(\vec{x}(t)) = 3\cos 4t + \cos 4t \sin 4t + 27t^3$$

$$\int_{\vec{x}} f\,ds = \int_0^{2\pi} (3\cos 4t + \underbrace{\cos 4t \sin 4t}_{\frac{1}{2}\sin 8t} + 27t^3) \cdot 5\,dt$$

$$= \left( \frac{3}{4} \sin 4t + \frac{-1}{16} \cos 8t + \frac{27}{4} t^4 \right) \cdot 5 \Big|_0^{2\pi}$$

$$= 0 + 0 + \frac{27 \cdot 5}{4} (2\pi)^4$$

$$= 540 \pi^4.$$

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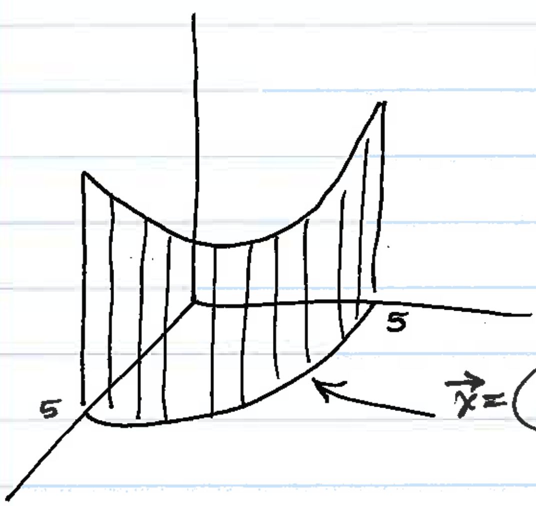
Exc # 34 p 427.

$h(x,y) = 10 - x - y$




$\vec{x} = (\underbrace{5\cos t}_{x}, \underbrace{5 \sin t}_{y}) \qquad 0 \le t \le \frac{\pi}{2}.$

$$\text{Area} = \int_{\vec{x}} h \, ds = \int_0^{\pi/2} (10 - 5\cos t - 5 \sin t) \cdot \underbrace{5 \cdot dt}_{ds}$$

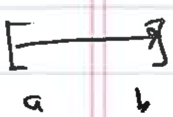
$$\vec{x}' = (-5 \sin t, 5 \cos t), \quad |x'| = 5$$

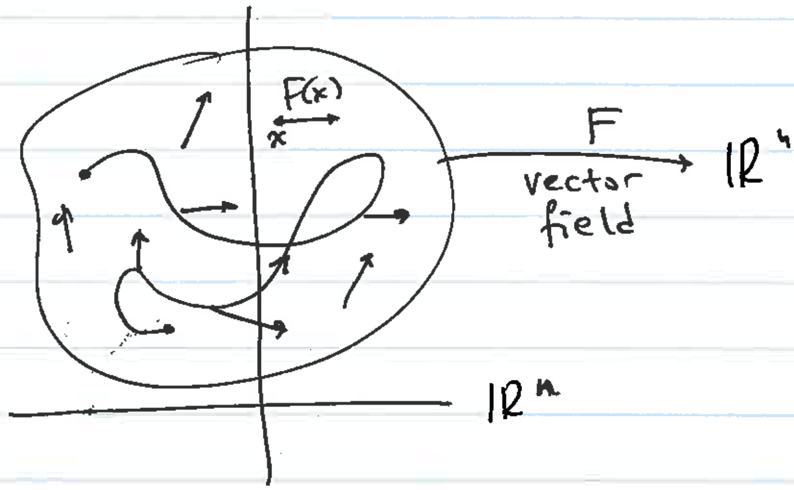
$$= 5 \cdot \left[ 10t - 5 \sin t + 5 \cos t \right] \Big|_0^{\pi/2}$$

$$= 5 \left\{ \left[ 5\pi - 5 \overbrace{\sin \tfrac{\pi}{2}}^{1} + 5 \overbrace{\cos \tfrac{\pi}{2}}^{0} \right] - \left[ 0 - 5 \overbrace{\sin 0}^{0} + 5 \overbrace{\cos 0}^{1} \right] \right\} = 25\pi - 5$$

## 6.1 II Vector Line integrals

<u>Defn</u> Let $\vec{x} : [a, b] \longrightarrow X \subseteq \mathbb{R}^n$ be a piecewise diff'ble curve




<u>One defines:</u>
$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \underbrace{\vec{x}'(t)\, dt}_{d\vec{s}}$$

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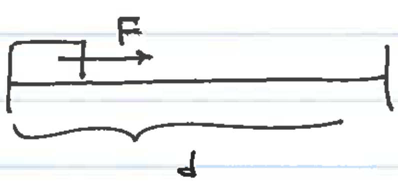
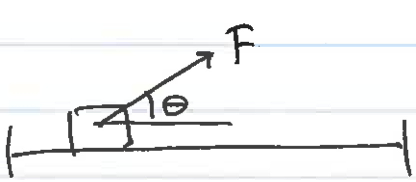
Compare to
$$\int f \cdot ds = \int_a^b f(\vec{x}(t)) \cdot \underbrace{\|\vec{x}'(t)\|\, dt}_{ds}$$

Motivation is from Physics

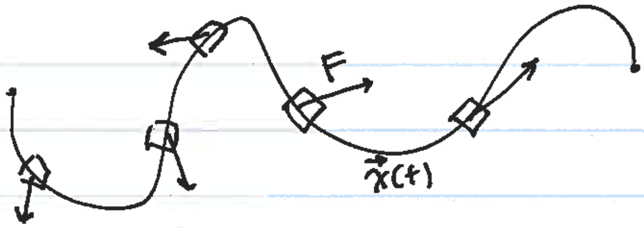



Work = F · d

products of real #'s




$W = F \cdot d \cdot \cos\theta$




$$W = \underbrace{\int_a^b \vec{F}(\vec{x}(t)) \cdot d\vec{s}}_{\underline{\text{work}}}$$

Another notation:

$$\int \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \underbrace{\vec{x}'(t)\, dt}_{d\vec{s}}$$

$$= \int_a^b \vec{F}(\vec{x}(t)) \cdot \underbrace{\frac{\vec{x}'(t)}{|\vec{x}'(t)|}}_{\substack{\vec{T} \\ \text{unit tangent}}} \cdot \underbrace{|\vec{x}'(t)|\, dt}_{ds}$$

$$= \int \vec{F} \cdot \vec{T}\, ds$$

$$\boxed{d\vec{s} = \vec{T}\, ds}$$

$$\vec{T}(t) = \frac{\vec{x}'(t)}{|\vec{x}'(t)|}$$ unit tangent vector (direction of the velocity)

6.1 Exc #8 p 426

$$F = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z)$$

$$\vec{x}(t) = (2t+1,\ t,\ 3t-1). \qquad 0 \le t \le 1.$$

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\,dt$$

$$= \int_0^1 (2t+1,\ t,\ 3t-1) \cdot (2,\ 1,\ 3)\,dt$$

$$= \int_0^1 (4t + 2 + t + 9t - 3)\,dt$$

$$= \int_0^1 (14t - 1)\,dt = 7t^2 - t \Big|_0^1 = 6.$$

Exc #12 $\quad \vec{F} = x\vec{i} + xy\vec{j} + xyz\vec{k} = (x, xy, xyz)$

$$\vec{x}(t) = (3\cos t, 3\sin t, 5t)$$

$$0 \le t \le 2\pi$$

$$\vec{F}(\vec{x}(t)) = (3\cos t, 9\cos t\sin t, 45t\cos t\sin t)$$

$$\vec{x}'(t) = (-3\sin t, 3\cos t, 5)$$

Find $\displaystyle\int \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t))\cdot\vec{x}'(t)\,dt$

$$= \int_0^{2\pi} \left(-9\cos t\sin t + 27\cos^2 t\cdot\sin t + 225\,t\cos t\sin t\right) dt$$

$u = \sin t \qquad v = \cos t, \quad dv = -\sin t$

$$= \underbrace{-\frac{9\sin^2 t}{2} + (-27)\frac{\cos^3 t}{3}\Bigg|_0^{2\pi}}_{0} + 225\int_0^{2\pi} t\underbrace{\cos t\sin t}_{\frac{1}{2}\sin 2t}\,dt$$

$$= 225\int_0^{2\pi} t\cdot\frac{1}{2}\sin 2t\,dt = \frac{-225\pi}{2} \quad \text{since:}$$

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$\displaystyle\int_0^{2\pi} t\sin 2t\,dt$ $\qquad$ Use By Parts:

$u = t \qquad du = dt$

$dv = \sin 2t\,dt \qquad v = -\frac{1}{2}\cos 2t$

$$= -\frac{1}{2}\cos 2t\cdot t\Bigg|_0^{2\pi} - \underbrace{\int_0^{2\pi} -\frac{1}{2}\cos 2t\,dt}_{0} = -\pi$$