April 17 ⓪

Plan for the rest of the semester

Apr	17	5.5
	19	3.3, 3.4
	20	6.1
	21	6.1
	24	6.2
	26	6.3
	27	qz 5, 6.3
	28	7.1
May	1	7.1, 7.2
	3	7.2
	4	7.3
	5	7.3

5.5 Continue

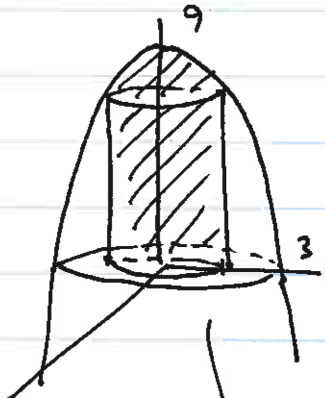
Exc # 40 p 373

Volume of the solid W bdd by

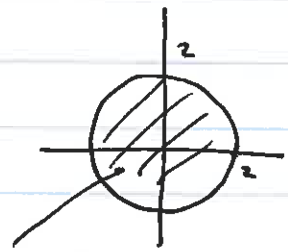
paraboloid $z = 9 - x^2 - y^2$

cylinder $x^2 + y^2 = 4$

xy-plane.



Top view



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$0 \le z \le 9 - r^2$

$$V = \iiint_W 1 \underset{dx\,dy\,dz}{dV} = \int_0^{2\pi} \int_0^2 \int_0^{9-r^2} 1 \cdot \underset{\text{Jacobian}}{r} \cdot dz \; dr \; d\theta$$

$$= \int_0^{2\pi} \int_0^2 rz \Big|_{z=0}^{z=9-r^2} dr \; d\theta$$

$$= \int_0^{2\pi} \int_0^2 r(9-r^2)\, dr \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 r(9-r^2)\, dr\right) = 2\pi \int_0^2 9r - r^3 \, dr$$

$$= 2\pi \cdot \left(\frac{9r^2}{2} - \frac{r^4}{4} \Big|_0^2 \right) = 2\pi \cdot (18 - 4) = 28\pi$$

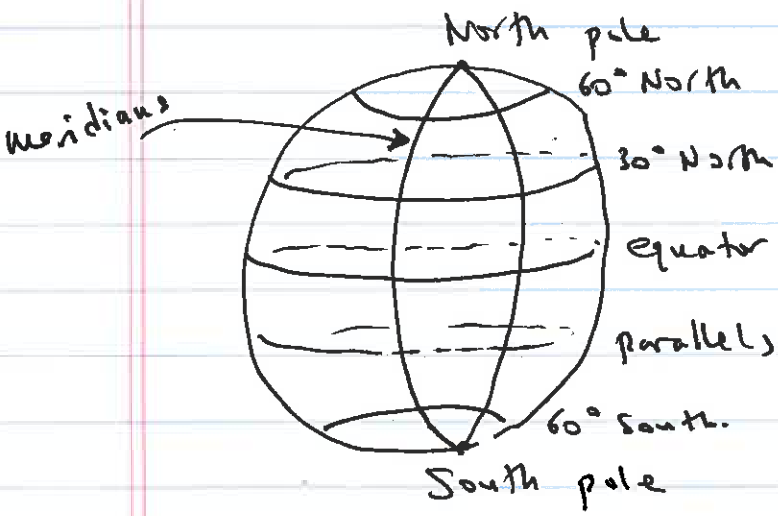
(b) If region W is filled with a substance of density $\delta(x,y,z) = x^2$,

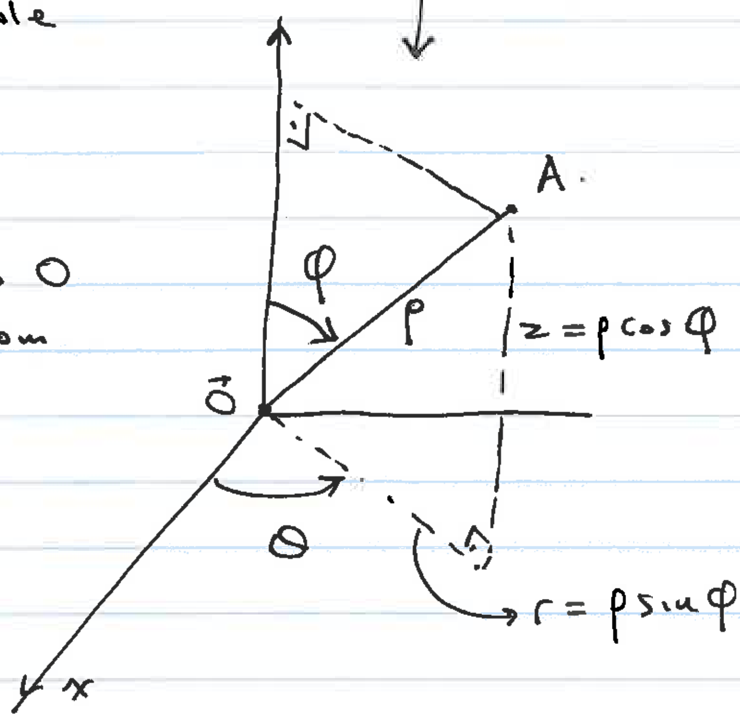
What is the total mass?

$$\underline{M} = \int_0^{2\pi} \int_0^2 \int_0^{9-r^2} (r\cos\theta)^2 \cdot r \cdot dz\,dr\,d\theta$$

$$= \cdots = \frac{76\pi}{3}.$$

Spherical Coordinates





ρ = distance from A to O

φ = angle of $\overline{OA}$ from + z-axis

θ

$$T^{-1}\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\dfrac{y}{x} + k\pi \\ \varphi = \cos^{-1}\dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$$

$$T\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

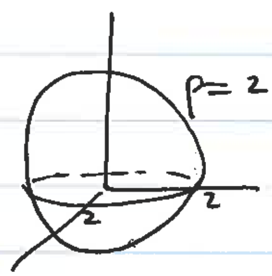
$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \pm\rho^2\sin\varphi$$

$$\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = dx\,dy\,dz.$$

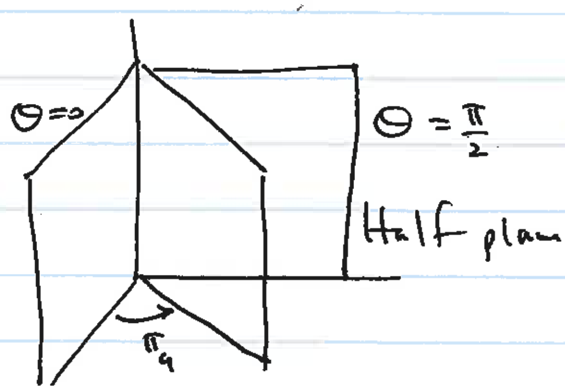
Basic surfaces

$\rho = 2$



sphere of radius 2

$\theta = \frac{\pi}{4}$



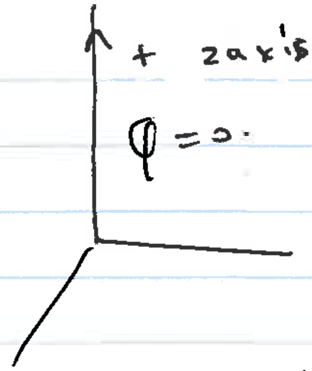
$\varphi = 0$

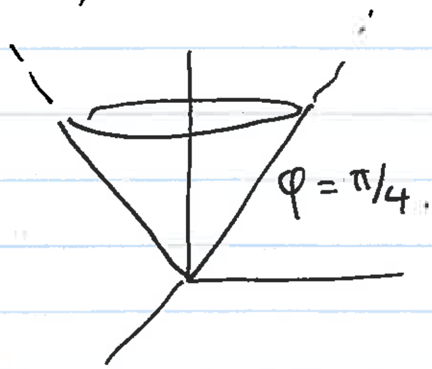
$\varphi = \frac{\pi}{4}$

$\varphi = \frac{\pi}{2}$

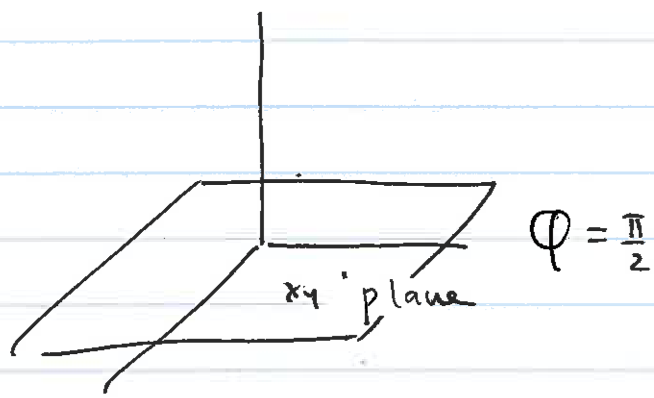
$\varphi = \frac{3\pi}{4}$

$\varphi = \pi$

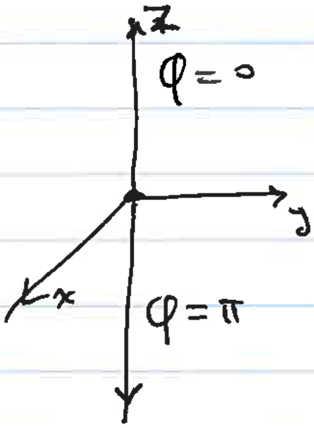


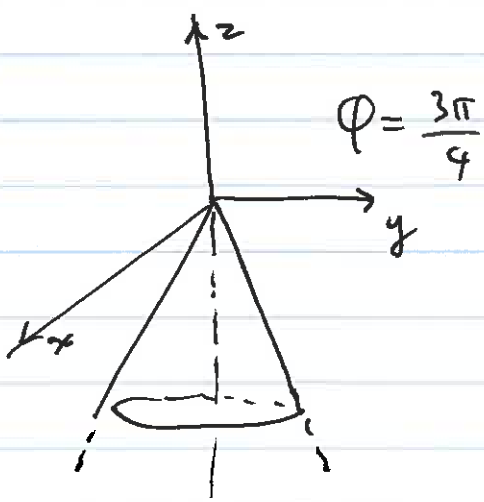


$z = x^2 + y^2$



$\varphi = \pi$



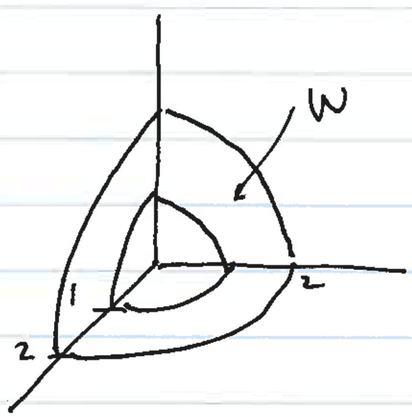


Ex 1 Let W be the region between the spheres

$$x^2+y^2+z^2=1$$
$$x^2+y^2+z^2=4$$

in the first octant,

Find
- Volume of W
- $\iiint_W x\,dV$



$$1 \le \rho \le 2$$
$$0 \le \theta \le \pi/2$$
$$0 \le \varphi \le \pi/2$$

$$V = \iiint 1 \; dx\,dy\,dz$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 1 \cdot \rho^2 \sin\varphi \cdot d\rho\, d\varphi\, d\theta$$

$$= \left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{\pi/2} \sin\varphi\, d\varphi\right)\left(\int_1^2 \rho^2\, d\rho\right)$$

$$= \frac{\pi}{2}\left(-\cos\varphi \Big|_0^{\pi/2}\right)\left(\frac{\rho^3}{3}\Big|_1^2\right)$$

$$= \frac{\pi}{2}\left(-\cos\frac{\pi}{2} + \cos 0\right)\left(\frac{8-1}{3}\right) = \frac{7\pi}{6}.$$

(b) $$\iiint_W x\, dV$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \underbrace{\rho \sin\varphi \cos\theta}_{x} \; \underbrace{\rho^2 \sin\varphi}_{\text{Jacobian}} \, d\rho \, d\varphi \, d\theta$$

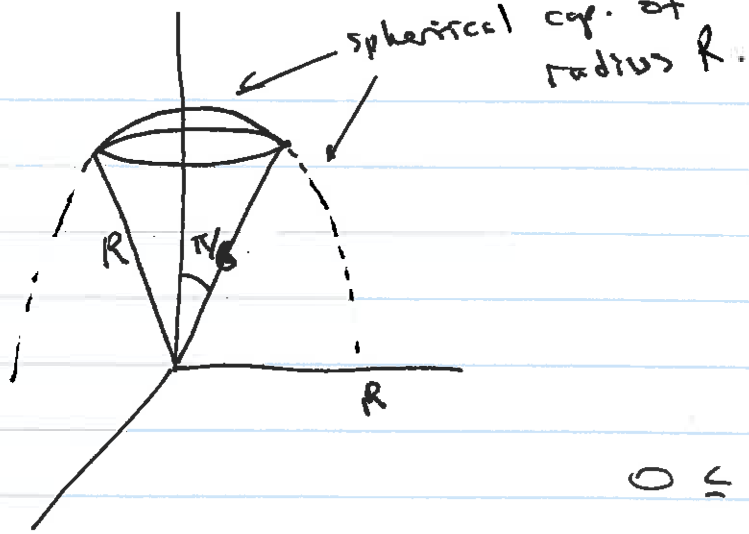
$$= \left(\int_0^{\pi/2} \cos\theta \, d\theta \right) \left(\int_0^{\pi/2} \sin^2\varphi \, d\varphi \right) \left(\int_1^2 \rho^3 \, d\rho \right)$$

$$= \left(\sin\theta \Big|_0^{\pi/2} \right) \int_0^{\pi/2} \frac{1 - \cos 2\varphi}{2} \, d\varphi \; \left(\frac{\rho^4}{4} \Big|_1^2 \right)$$

$$= (1)\left(\frac{\varphi}{2} - \frac{\sin 2\varphi}{4} \Big|_0^{\pi/2} \right) \left(\frac{16-1}{4} \right)$$

$$= 1 \cdot \left(\frac{\pi}{4} \right) \left(\frac{15}{4} \right) = \frac{15\pi}{16}.$$

Ex 2



$0 \le \rho \le R.$
$0 \le \theta \le 2\pi$
$0 \le \varphi \le \frac{\pi}{6}.$

Volume

$$V = \int_0^{2\pi}\int_0^{\pi/6}\int_0^R 1 \cdot \rho^2 \sin\varphi \cdot d\rho\, d\varphi\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/6} \sin\varphi\, d\varphi\right)\left(\int_0^R \rho^2\, d\rho\right)$$

$$= 2\pi \cdot \left(-\cos\varphi \Big|_0^{\pi/6}\right) \frac{R^3}{3}.$$

$$= \frac{2\pi}{3}R^3\left(-\cos\frac{\pi}{6} + \cos 0\right)$$

$$= \frac{2\pi}{3}R^3\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^R \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \frac{4\pi}{3}R^3$$ = volume of a ball of radius R.