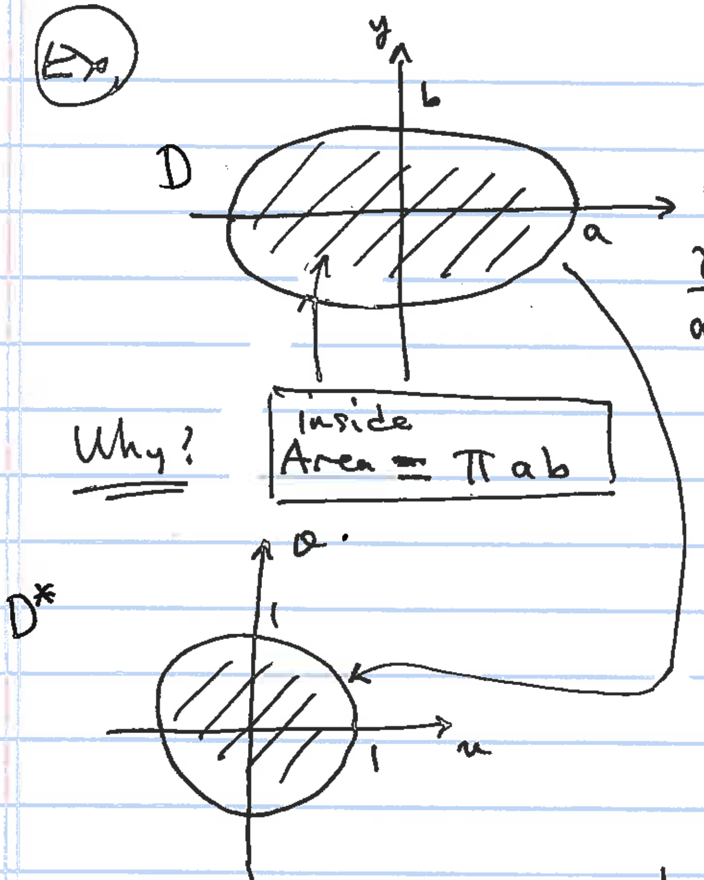
5.5 Continue

1

Ex,



Why? $\boxed{\text{inside Area} = \pi ab}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad a,b>0$$

$$T(x,y)=\left(\frac{x}{a},\frac{y}{b}\right)=(u,v)$$

$$DT=\begin{bmatrix}\frac{1}{a} & 0\\ 0 & \frac{1}{b}\end{bmatrix}=$$

$$\det DT=\frac{1}{ab}=\frac{\partial(u,v)}{\partial(x,y)}$$

$$du\,dv=\left|\frac{\partial(u,v)}{\partial(x,y)}\right|dx\,dy=\frac{1}{ab}\,dx\,dy$$

$$ab\,du\,dv=dx\,dy$$

Area of Ellipse (the inside) $\displaystyle =\iint_D 1\,dx\,dy=\iint_{D^*}1\cdot ab\,du\,dv$

$$=ab\iint_{D^*}1\,du\,dv=ab\,\underbrace{\text{area}(D^*)}_{\pi}=\pi ab.$$

Circumference $\longrightarrow$ an integral formula / Elliptic integral

Discussion:

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$T(u,v) = (x,y)$$
$$(u,v) = T^{-1}(x,y)$$

$\Bigg\}$ T coordinate transformation
T is 1-1, onto
$\det DT \neq 0$

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$$

requires: $\dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(x,y)} = 1.$ $\quad n=2$

Reason

Th.

$$T^{-1}(T(\vec{x})) = Id(\vec{x})$$

Chain Rule

$$DT^{-1}(T(x)) \cdot DT(x) = DI(\vec{x}) = I_n.$$

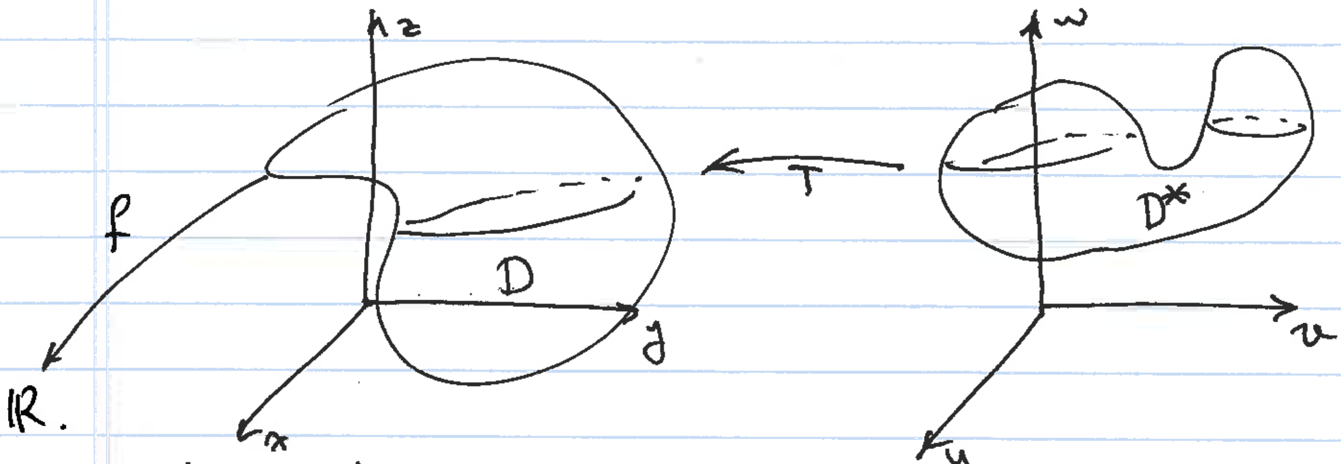
Identity matrix.

det

$$\left((\det DT^{-1})(T(x))\right) \cdot \det DT(x) = 1$$

Thm: (Jacobi) $n=3$.

Let



Assume that:

$T(u,v,w)=(x,y,z)$

T is a coordinate transformation from D^* ont D,

T is 1-1

T is continuously diff'ble

$\det DT \neq 0$

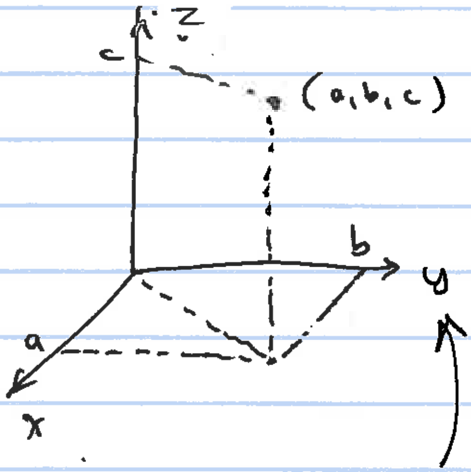
$f: D \to \mathbb{R}$ continuous.

Then:

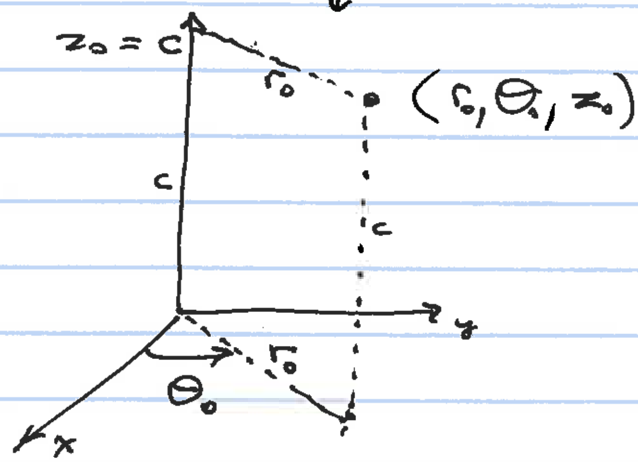
$$\iiint_D f(x,y,z)\, \underbrace{dV_{xyz}}_{dx\,dy\,dz} = \iiint_{D^*} f(T(u,v,w))\,|\det DT|\, dV_{uvw}$$

$$\det DT = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

CYLINDRICAL COORDINATES



Rectangular Coordinates



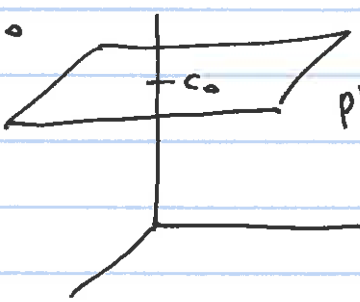
$$(x, y, z) \underset{T^{-1}}{\overset{T}{\rightleftarrows}} (r, \theta, z)$$

$$T^{-1}: \begin{cases} r = \sqrt{x^2+y^2} & r \geq 0 \\ \theta = \tan^{-1}\dfrac{y}{x} + k\pi & 0 \leq \theta \leq 2\pi \\ z = z & z \in \mathbb{R}. \end{cases}$$

$$T: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

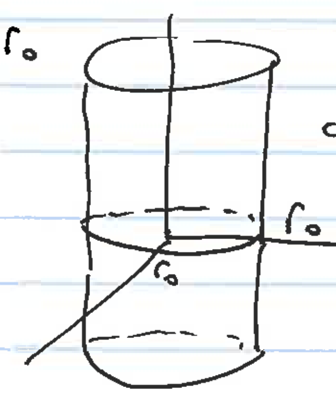
Basic surfaces

$z = c_0$

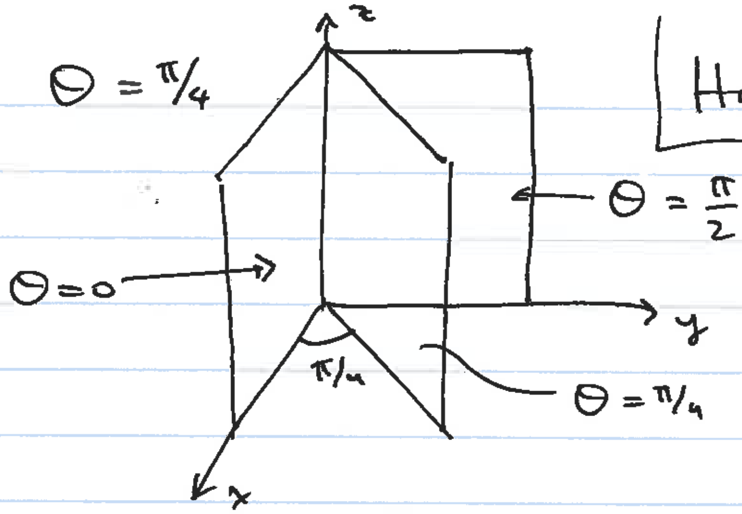


plane

$r = r_0$



cylinder

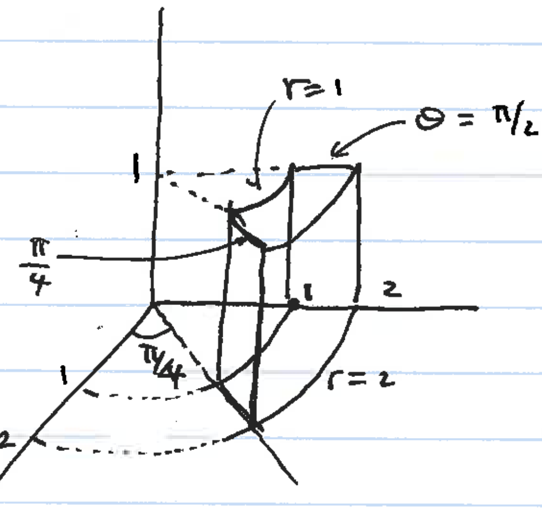


$\theta = \theta_0$

Half planes

Ex

$$1 \le r \le 2$$
$$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$$
$$0 \le z \le 1$$



Jacobian:

$$T = \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad DT = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det DT = r\cos^2\theta + r\sin^2\theta = r$$

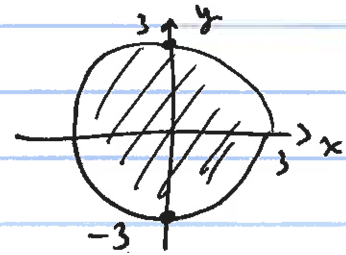
$$dx\,dy\,dz = \underbrace{\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right|}_{r \quad \text{Jacobian}} dr\,d\theta\,dz$$

5.5 Exc #28 p 372.

$$I = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} \frac{e^z}{\sqrt{x^2+y^2}} \, dz\, dy\, dx.$$

Convert to cylindrical Coordinates & evaluate

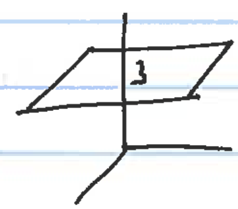
$$\left.\begin{array}{l} -3 \le x \le 3 \\ -\sqrt{9-x^2} \le y \le \sqrt{9-x^2} \end{array}\right\} \longrightarrow$$

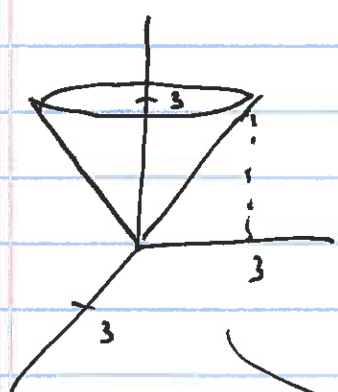


Top view

$$\underbrace{\sqrt{x^2+y^2} \le z \le 3}$$

$z = 3$

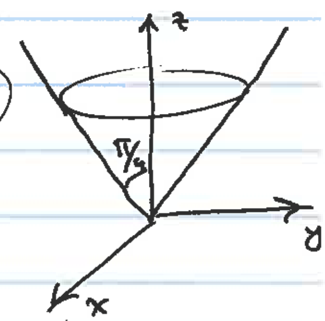




$z = \sqrt{x^2+y^2}$

$\Updownarrow$

$z = r$



$$\left.\begin{array}{l} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{array}\right\}$$ Top view: same

$$r \le z \le 3$$

$|\det DT|$ Jacobian

$$I = \int_0^{2\pi} \int_0^{3} \int_r^{3} \frac{e^z}{r} \cdot r \cdot dz\, dr\, d\theta$$

$$I = \int_0^{2\pi} \int_0^3 \int_r^3 e^z \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 e^z \Big|_{z=r}^{z=3} \, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 (e^3 - e^r)\, dr\, d\theta$$

$$= \int_0^{2\pi} re^3 - e^r \Big|_{r=0}^{r=3} \, d\theta$$

$$= \int_0^{2\pi} (3e^3 - e^3) - (0 - e^0) \, d\theta$$

$$= \int_0^{2\pi} (2e^3 + 1)\, d\theta$$

$$= 2\pi \cdot (2e^3 + 1)$$

Caution this integral is improper since

$\frac{e^z}{\sqrt{x^2+y^2}} = \frac{e^z}{r} = f(r, \theta, z)$ is undefined along $r=0$ line (z-axis)

However the integral converges by calculating for $\varepsilon \le r \le 3$ and by taking limit $\varepsilon \to 0$.