MATH 2850 **Practice questions for Midterm 2.**

Your test is 50 minutes long. All of these 17 questions should be doable in 3 hours, so in average each should take 10 minutes. (Caution some are 5-minute and some are 15-minute problems.)

Problem 1. Let $f(x,y) = x^3 - 3x^2 + y^3 - 27y : \mathbf{R}^2 \to \mathbf{R}$.

a. Find all critical points of f.

b. Find all of the local maxima, local minima, and saddle points of f.

Problem 2. a. Evaluate $\int_0^2 \int_0^{2x} \int_0^{3y} xyz \, dz \, dy \, dx$.

b. Sketch the domain of integration of the integral above.

Problem 3. Consider the integral $\int_0^3 \int_{x^2}^9 4xe^{(y^2)} \, dy \, dx$.

a. Sketch the domain of integration.

b. Reverse the order of integration.

c. Find the (numerical) value of the integral above.

Problem 4. Find the maximum and minimum values of $f(x,y) = x^2 + xy + y^2$ on the domain defined by $g(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 \leq 1$. Indicate any theorem you use in your solution, where and how it is used.

Problem 5. Let $\mathbf{X}(t) = (4\cos t, 4\sin t, 3t)$ for $0 \leq t \leq 2\pi$.

a. For the parametric motion defined by $\mathbf{X}(t)$, find its velocity, acceleration and speed.

b. Find a parametric equation for the tangent line to $\mathbf{X}(t)$ when $t = \pi$.

c. Find the length of $\mathbf{X}(t)$.

d. Sketch the curve $\mathbf{X}(t)$, and include the coordinates of its end points.

Problem 6. Find the first order Taylor polynomial for $f(x,y) = x^2\cos 3y + xy^2$ near $(a,b) = (3,0)$.

b.Use part (a) to approximate $f(3.04, 0.03)$.

c. Find the second order Taylor polynomial for f near $(a,b) = (3,0)$.

Problem 7 a. Sketch the domain of integration of the integral and reverse the order of integration: $\int_0^1 \int_0^{2x} e^x \cos y \, dy \, dx$.

b. Sketch the domain of integration of the integral $\int_0^{\pi/2} \int_y^{4-y} x \sin y \, dx \, dy$ and calculate the value of the integral.

Problem 8. Let $h(x,y) = x^2 - y^2 - 2y$.

a. Find all critical points of $h(x,y)$ on $\mathbf{R}^2$. Determine whether each critical point is a relative maximum, minimum, or saddle point.

b. Find the points at which the function $h(x,y) = x^2 - y^2 - 2y$ attains its maximum and minimum values on the circle $x^2 + y^2 = 1$. State the maximum and minimum values.

Problem 9.

Let $f(x,y,z) = x^2y^3 - yz + 2xz$

a. Calculate ∇f.

b. Calculate the directional derivative of f at $(x,y,z) = (1,0,2)$ in the direction of the vector $(2,-2,1)$.

c. In which direction does f decrease fastest at $(1,0,2)$?

d. Find an equation for tangent plane to the level set $x^2y^3 - yz + 2xz = 4$ at $(1,0,2)$.

Problem 10. Find all critical points of $f(x,y) = x^3 - 2xy + y^2 - x : \mathbf{R}^2 \to \mathbf{R}$, and determine whether each critical point is a local maximum, a local minimum or a saddle point by using the Second Derivative Test.

Problem11. a. Why do the equations $\begin{cases} x^2 + y^2 + z^2 - w = 1 \\ x - y + z + w^2 = 5 \end{cases}$ implicitly define x and y as functions of z and w near the point $(x,y,z,w) = (1,1,1,2) = \mathbf{p}$?

b. If $g(z,w) = (x,y)$ is the solution function near $\mathbf{p}$, compute the derivative matrix g' near $\mathbf{p}$ in terms of x,y,z and w.

Problem 12. For each of the following iterated integrals, sketch the domain of integration of the integral and find the value of the integral.

a. $\int_0^2 dx \int_{2x}^4 dy \int_0^y e^z dz$

b. $\int_0^2 dx \int_{2x}^4 \cos(y^2) dy$

Problem 13. Let B be the subset of $\mathbf{R}^3$, defined by
$y^2 \geq z$,
$x + y \leq 1$,
$x \geq 0$,
$y \geq 0$, and
$z \geq 0$.
a. Sketch the region B.
b. Calculate the volume of B.
c. Calculate $\iiint_B 12yz\, dV$

Problem14. Sketch the region D defined by $x^2 + y^2 \leq 9, x \geq 0$ and $y \geq 0$ in $\mathbf{R}^2$, describe the region D by using polar coordinates, and calculate $\iint\limits_D \sqrt{x^2 + y^2}\ dx\, dy$.

Problem 15. Let a transformation T from the uv-plane to the xy-plane be defined by $x = u + v$ and $y = u - v$. Let R_{uv} be the rectangular region given by $0 \leq u \leq 2$, and $0 \leq v \leq 1$ in the uv-plane.
a. Find and sketch the region $R_{xy} = T(R_{uv})$, the image of R_{uv} under the transformation T.
b. Find $\frac{\partial(x,y)}{\partial(u,v)}$.
c. Transform $\int_{R_{xy}} xy\, dx\, dy$ to an integral over R_{uv}. Write the integral as an iterated integral, but do not evaluate it.

Problem 16. For the curve L parametrized by $g(t) = (2t, \ln t, t^2)$, for $1 \leq t \leq e^2$, calculate the following.
a. the velocity
b. the speed
c. The length of L
d. The equation of the tangent line to L when $t = e$.

Problem 17. Let $S = \{(x,y) : 2y = x^2\}$ and $(0,a)$ be a given point on the $y - axis$ in $\mathbf{R}^2$. Find the closest point(s) of S to $(0,a)$ in terms of a.
HINTS: It is easier to minimize the square of the distance function from $(0,a)$. Also, you will have different types of answers depending on whether a is large or small.

ordered According to
the Practice test numbers

4/12/17 (1)

#5) a) $X(t) = (4\cos t,\ 4\sin t,\ 3t)$ $\quad 0 \le t \le 2\pi$

$X'(t) = (-4\sin t,\ 4\cos t,\ 3)$ velocity

$X''(t) = (-4\cos t,\ -4\sin t,\ 0)$ acceleration

$|X'(t)| = \sqrt{16\sin^2 t + 16\cos^2 t + 9}$ speed
$= 5$

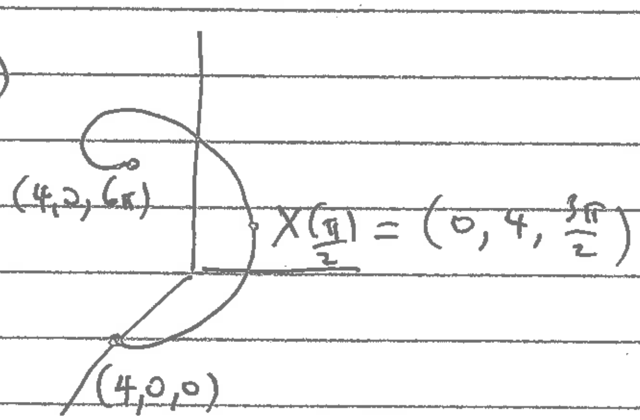
b) Tangent line at $t = \pi$

$X(\pi) = (-4,\ 0,\ 3\pi)$.
$X'(\pi) = (0,\ -4,\ 3)$

$l(s) = (-4, 0, 3\pi) + s(0, -4, 3)$

c) $l = \int_0^a |X'(t)|\,dt = \int_0^{2\pi} 5\,dt = 10\pi$.

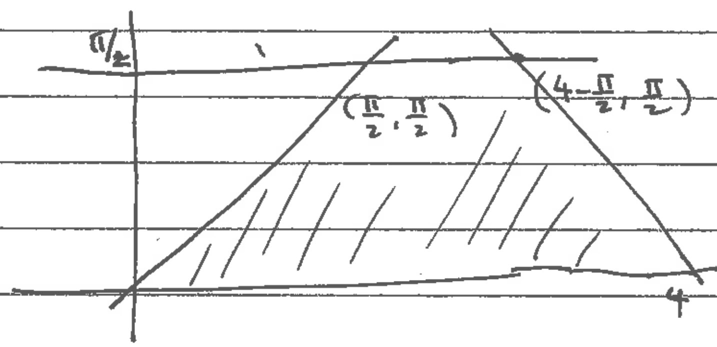
d)



#7b) $$\int_0^{\pi/2}\int_y^{4-y} x\sin y\,dx\,dy$$

$0 \le y \le \frac{\pi}{2}$

$y \le x \le 4-y$



$x = 4-y$

$x+y=4$

$$\int_0^{\pi/2}\int_y^{4-y} x\sin y\,dx\,dy$$

$$= \int_0^{\pi/2} \frac{1}{2}x^2\sin y\Big|_{x=y}^{x=4-y}\,dy$$

$$= \int_0^{\pi/2} \frac{1}{2}\Big(\underbrace{(4-y)^2 - y^2}_{16-8y+\cancel{y^2}-\cancel{y^2}}\Big)\cdot\sin y\,dy$$

By Parts

$\sin y\,dy = dv$

$8-4y = u$

$$I = \int_0^{\pi/2}(8-4y)\sin y\,dy$$

$\sin y \, dy = dv \qquad v = -\cos y$

$8 - 4y = u \qquad du = -4dy$

$$I = (8-4y)(-\cos y)\Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos y)(-4\,dy)$$

$$= (8-2\pi)\underbrace{(-\cos\tfrac{\pi}{2})}_{0} - 8(-\cos 0) - \int_0^{\pi/2} 4\cos y \, dy$$

$$= 8 - 4\sin y\Big|_0^{\pi/2} = 8 - (4\underbrace{\sin\tfrac{\pi}{2}}_{1} - 4\underbrace{\sin 0}_{0})$$

$$= 4.$$

#9 a) $\nabla f = (2xy^3 + 2z,\ 3x^2y^2 - z,\ -y + 2x)$

$D_u f(a) = \nabla f(a) \cdot u.$

$\nabla f(1,0,2) = (4, -2, 2)$

$u = \frac{(2, -2, +1)}{\sqrt{4+4+1}} = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$

b) $(D_u f)(a) = (4, -2, 2) \cdot \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$

$= \frac{1}{3}(8 + 4 + 2) = \frac{14}{3}$

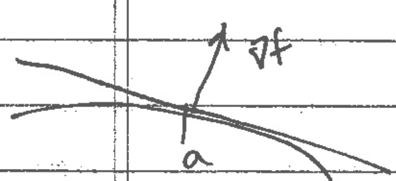
c) $\frac{-\nabla f(a)}{|\nabla f(a)|} = -\frac{(4, -2, 2)}{\sqrt{16+4+4}} = -\frac{(4, -2, 2)}{2\sqrt{6}}$

$= \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$

d)

$(\vec{x} - (a)) \cdot \nabla f(a) = 0$

$((x, y, z) - (1, 0, 2)) \cdot (4, -2, 2) = 0$



$4x - 2y + 2z = 4 + 4 = 8$

$\boxed{2x - y + z = 4}$

#11

$$\begin{array}{cccc} x & y & z & w \\ \end{array}$$

$$\begin{bmatrix} 2x & 2y & 2z & -1 \\ 1 & -1 & 1 & 2w \end{bmatrix} (1,1,1,2)$$

$$= \begin{bmatrix} \underbrace{2 \quad 2}_{\text{solving for } x \,\&\, y} & 2 & -1 \\ 1 \quad -1 & 1 & 4 \end{bmatrix} \qquad (x,y) = g(z,w)$$

$$\begin{vmatrix} 2 & 2 \\ 1 & -1 \end{vmatrix} = -4 \neq 0 \implies \exists \text{ such } g(z,w) = (x$$

$$g'(\underbrace{1,1,1,2}_{\text{at } p}) = - \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

$$= - \frac{1}{-4} \begin{bmatrix} -1 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -4 & -7 \\ 0 & 9 \end{bmatrix}.$$

near p:

$$g' = - \begin{bmatrix} 2x & 2y \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2z & -1 \\ 1 & 2w \end{bmatrix}$$

$$= - \frac{1}{-2x-2y} \begin{bmatrix} -1 & -2y \\ -1 & 2x \end{bmatrix} \begin{bmatrix} 2z & -1 \\ 1 & 2w \end{bmatrix}$$

$$= \frac{1}{2(x+y)} \begin{bmatrix} -2z-2y & 1-4yw \\ -2z+2x & 1+4xw \end{bmatrix} \qquad \begin{array}{l} x=y=z=1 \\ w=2 \end{array}$$

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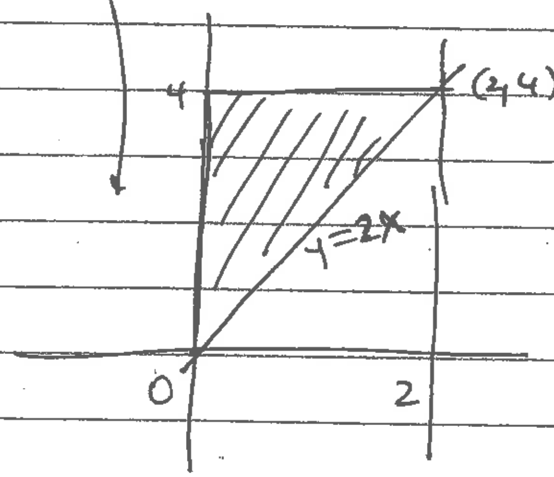
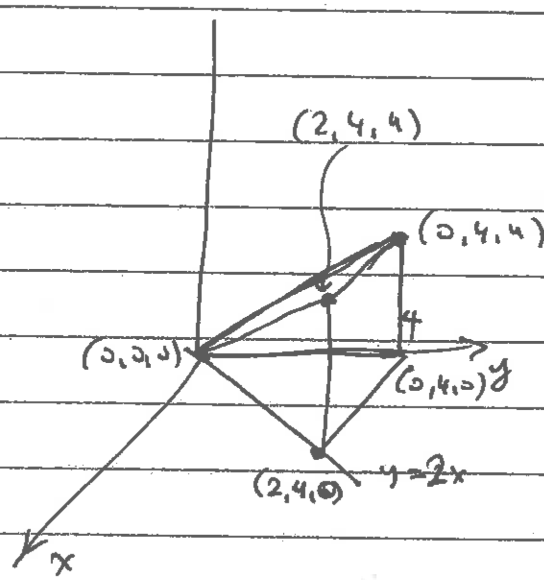
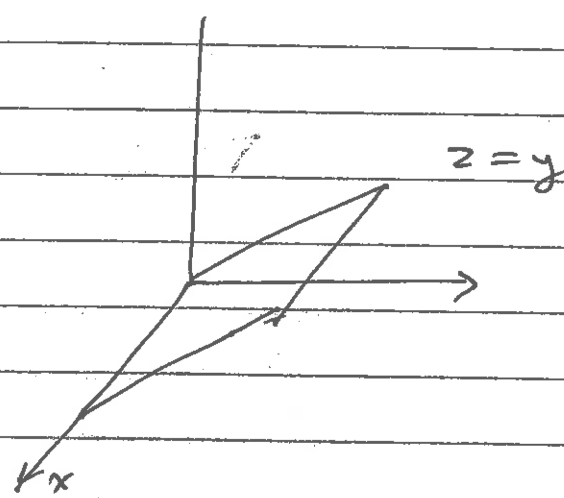
12 (a) $\int_0^2 dx \int_{2x}^4 dy \int_0^y e^z dz = \int_0^2 \int_{2x}^4 \int_0^y e^z \, dz \, dy \, d$

$0 \le x \le 2$
$2x \le y \le 4$
$0 \le z \le y$





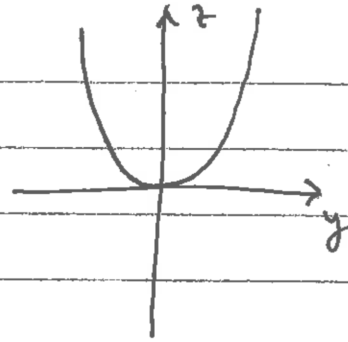


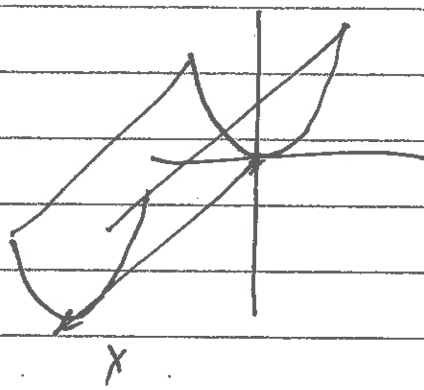
$y^2 \geq z$

$x+y \leq 1$

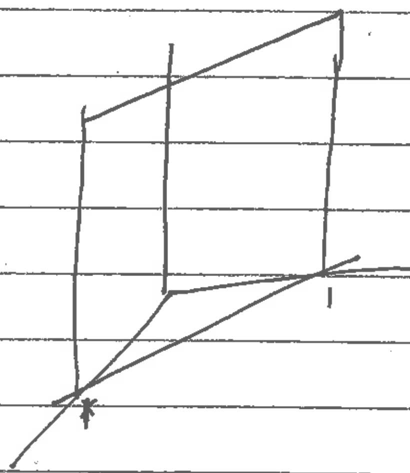
$x, y, z \geq 0.$

#13 1) $y^2 = z$

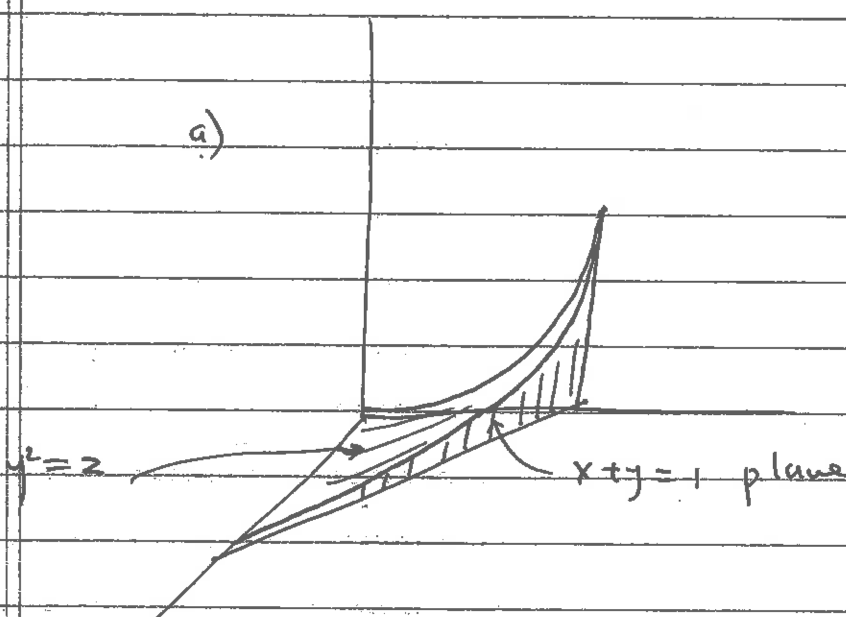




2) $x+y=1$



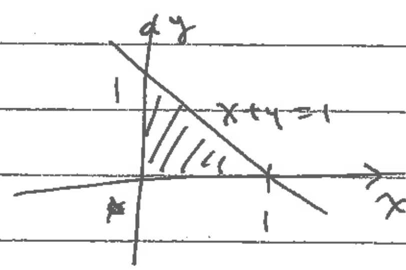
a)



$0 \leq y \leq 1$

$0 \leq x \leq 1-y$

b) Top view



$0 \leq x \leq 1$ I

$0 \leq y \leq 1-x$

$y^2 = z$

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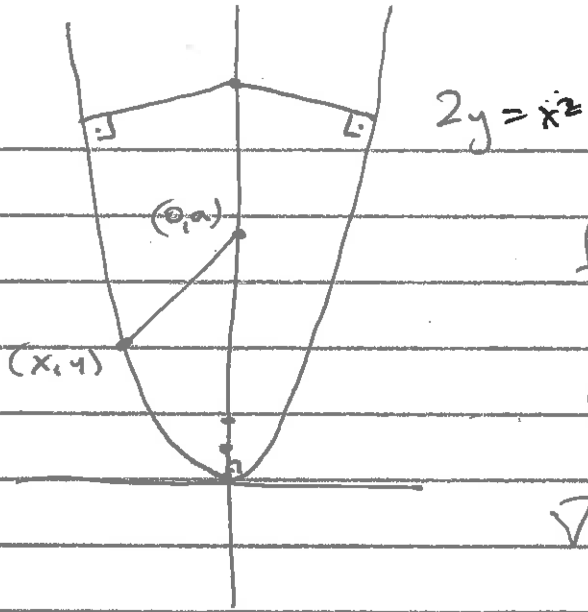
b) Volume $\int_0^1 \int_0^{1-x} \int_0^{y^2} 1 \cdot dz\, dy\, dx$

c) $\int_0^1 \int_0^{1-x} \int_0^{y^2} 12yz\, dz\, dy\, dx$

#17



$$f = (x-0)^2 + (y-a)^2$$

$$g = x^2 - 2y$$

$$\nabla f = \lambda \nabla g$$

$$\nabla f = (2x,\ 2(y-a))$$
$$\nabla g = (2x,\ -2)$$

① $2x = 2x\lambda$

② $2(y-a) = -2\lambda.$

③ $2y = x^2$

① $2x - 2x\lambda = 0$

$2x(1-\lambda) = 0$

$x = 0$ | $\lambda = 1.$

③ $y = 0$ | $2(y-a) = -2$

$y - a = -1$

$0 \le y = a - 1$

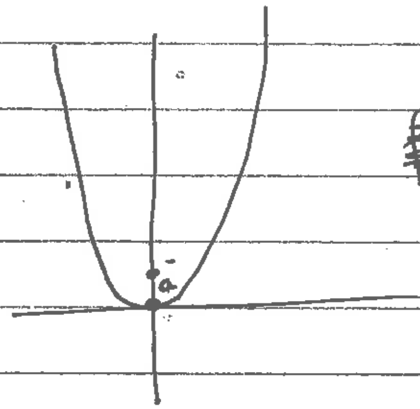
$a < 1$ no soln | $a \geq 1$

$2y = x^2$

$2(a-1) = x^2$

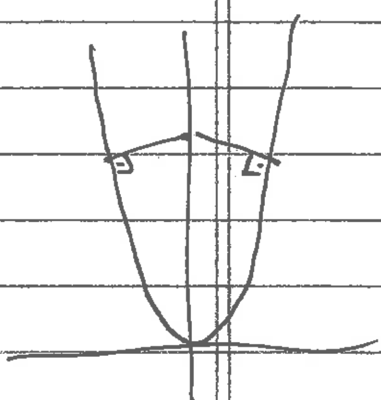
$x = \pm\sqrt{2(a-1)}.$

Case 1 $a < 1$, only one c.p. $(0,0)$



	$x^2 + (y-a)^2$
$(0,0)$ closest	a^2 distance squared

Case 2 $a \geq 1$



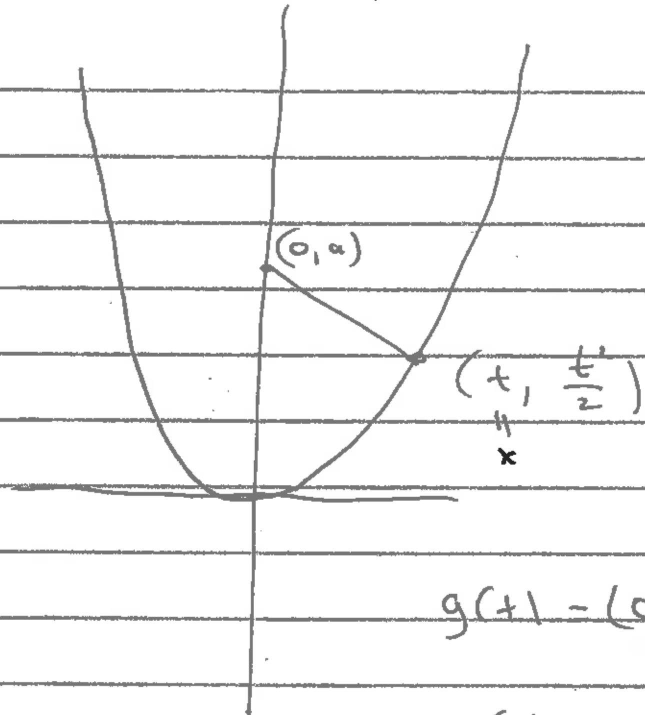
	$x^2 + (y-a)^2$
$(0,0)$	a^2
$(\sqrt{2(a-1)}, a-1)$	$2a-1 \leftarrow$ smaller when $a>1$
$(-\sqrt{2(a-1)}, a-1)$	

$$x^2 + (y-a)^2 = \left(\pm\sqrt{2(a-1)}\right)^2 + (\cancel{a} - 1 - \cancel{a})^2$$

$$= 2a - 2 + 1$$

$$= 2a - 1$$

Another method via parametrization



$2y = x^2$

$$g(t) = (0-t)^2 + \left(a - \frac{t^2}{2}\right)^2.$$

$$g(t) = t^2 + a^2 - at^2 + \frac{t^4}{4}$$

$$-\infty < t < \infty.$$

$$g'(t) = 2t - 2at + t^3 \stackrel{\cdot}{=} 0$$

$$= \underset{\sim}{t}\,(\underbrace{2 - 2a + t^2})$$

$t = 0$ OR. $2 - 2a + t^2 = 0$ $\quad a \geq 1$

$t^2 = 2a - 2$

$t = \pm\sqrt{2(a-1)}$ $\quad a \geq 1$

$t =$ no soln $\quad a < 1$

$a < 1$

	g
$(0,0)$	a^2

$a \geq 1$

$(0,0)$	a^2
$(\pm\sqrt{2(a-1)}, a-1)$	$2a-1$

From the textbook

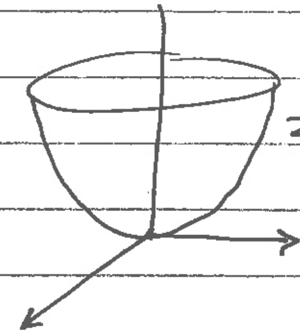
5.[illegible] p348 #16

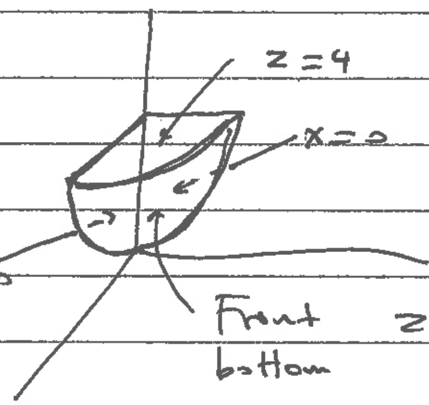
$z = x^2 + y^2$

$x = 0$

$y = 0$

$z = 4$



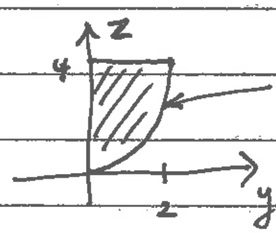


$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} 3x \; dz\, dy\, dx$$

solve for x

Top view

view x-axis direction



$z = y^2$ (in yz plane $x = 0$)

$$\int_0^2 \int_{y^2}^{4} \int_0^{\sqrt{z-y^2}} 3x \; dx\, dz\, dy$$

$0 \le y \le 2$

$y^2 \le z \le 4$

$0 \le x \le \sqrt{z-}$