April 10

①

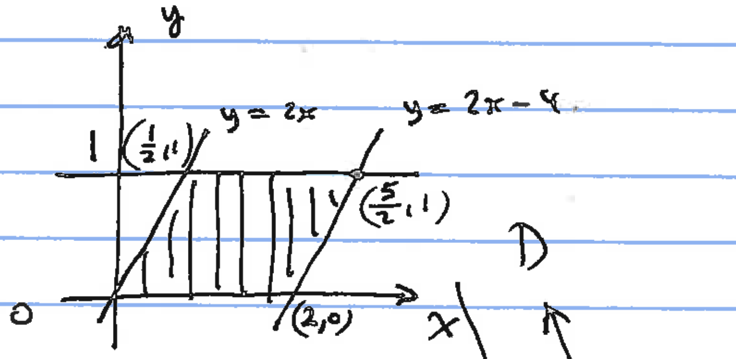
5.5 $n=2$, Examples.

Exc #8 5.5 p 371

$$I = \int_0^1 \int_{y/2}^{(y/2)+2} (2x-y)\,dx\,dy$$

$0 \le y \le 1$

$\frac{y}{2} \le x \le \frac{y}{2} + 2$



$x = \frac{y}{2} \iff y = 2x$

$x = \frac{y}{2} + 2$

$2x - 4 = y$

(b) $u = 2x - y$
$v = y$ } T substitution

$T(x,y) = (2x-y, y) = (u,v)$

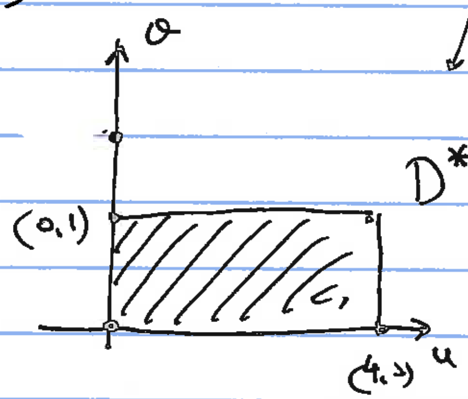
T linear

$T(0,0) = (0,0)$

$T(2,0) = (4,0)$

$T(\frac{1}{2},1) = (0,1)$

$T(\frac{5}{2},1) = (4,1)$



$$I = \int_0^4 \int_0^1 u \quad \frac{1}{2} \quad dv\,du$$

$$T(x,y) = (2x-y\ ,\ y) = (u,v)$$

$$(\quad = DT = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \qquad \det DT = 2$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 2$$

$$du\,dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$$

$$du\,dv = 2\,dx\,dy \implies dx\,dy = \tfrac{1}{2}\,du\,dv$$

$$I = \int_0^4 \int_0^1 u\,\tfrac{1}{2}\,dv\,du$$

$$= \left(\int_0^4 u\,du\right)\left(\int_0^1 \tfrac{1}{2}\,dv\right)$$

$$= \frac{u^2}{2}\Big|_0^4 \cdot \tfrac{1}{2}v\Big|_0^1 = 8 \cdot \tfrac{1}{2} = 4$$

TRICK:

$$\int_a^b \int_c^d f(x)\,g(y)\,dy\,dx = \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy$$

* all constant bounds

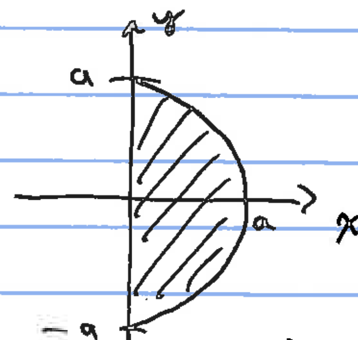
Exc # 16 p 372.

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} e^{x^2+y^2}\,dx\,dy$$

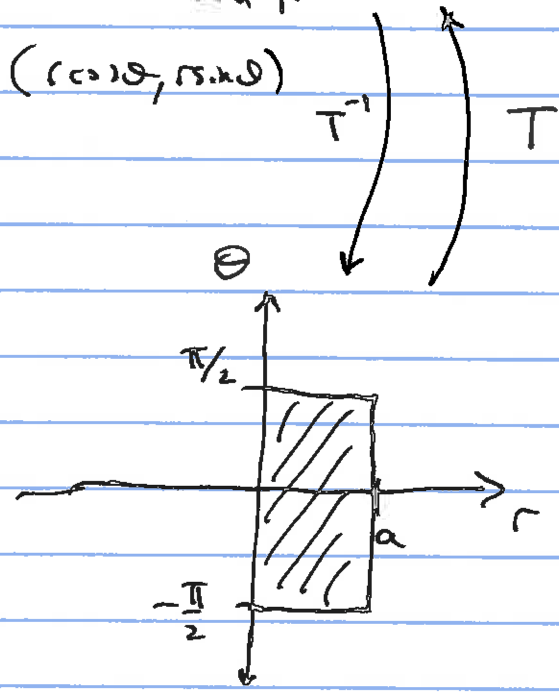
Cannot do it directly. Use Polar Coordinates

$-a \le y \le a$

$0 \le x \le \sqrt{a^2-y^2}$



$(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$



$\boxed{r\,dr\,d\theta = dx\,dy}$

since $|r| = r \ge 0$

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{a} e^{r^2}\cdot r\cdot dr\,d\theta$$

$u = r^2$
$du = 2r\,dr$

$$= \left(\int_0^a e^{r^2} r\,dr\right)\left(\int_{-\pi/2}^{\pi/2} d\theta\right) = \int_0^{a^2}\frac{1}{2}e^u\,du\cdot\pi = \frac{\pi}{2}\left(e^{a^2}-1\right).$$

Ex.

Calculate $\displaystyle\iint_D y^2 e^{y^2}\,dy\,dx$

where D is the region bdd by

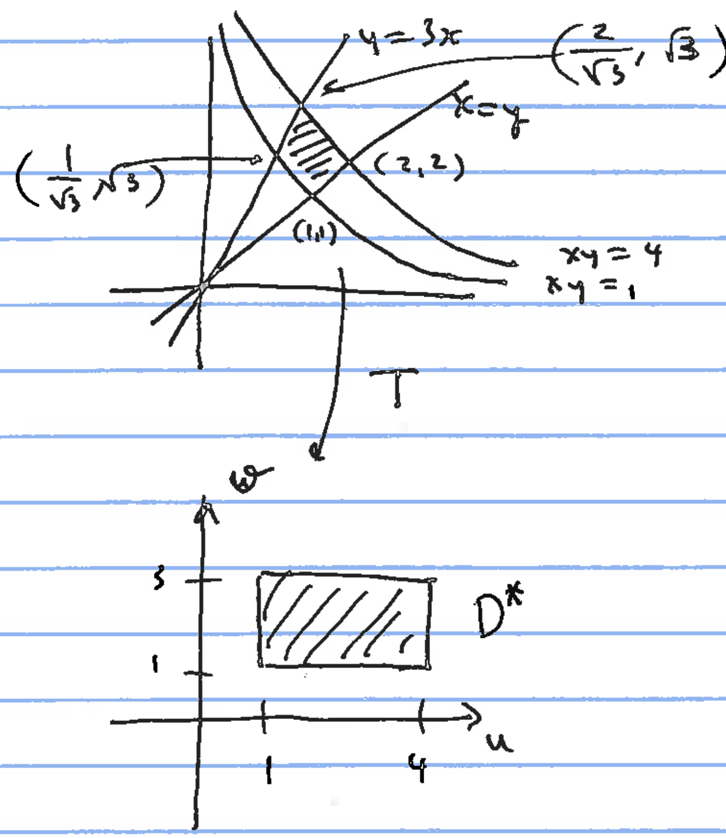
$y = x$

$y = 3x$

$xy = 4$

$xy = 1$

$x, y \geq 0$



$u = xy$

$v = \dfrac{y}{x}$

$1 \leq xy = u \leq 4$

$1 \leq \dfrac{y}{x} = v \leq 3$

$T(x,y) = \left(xy, \dfrac{y}{x}\right) = (u, v)$

$\det DT = \dfrac{\partial(u,v)}{\partial(x,y)}$; $\quad DT = \begin{bmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det DT = \left(y \cdot \frac{1}{x}\right) - \left(-\frac{y}{x^2} \cdot x\right)$$

$$= \frac{2y}{x} > 0$$

$$I = \iint_D y^2 e^{y^2}\, dy\, dx = \int_1^4 \int_1^3 e^{uv} \cdot uv \cdot \underbrace{\frac{1}{2v} \cdot dv\, du}_{dy\,dx}$$

$$u = xy$$
$$v = \frac{y}{x} \qquad y^2 = uv$$

$$du\, dv = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\, dy$$

$$du\, dv = \frac{2y}{x}\, dx\, dy$$

$$\frac{1}{2v}\, du\, dv = \frac{x}{2y}\, du\, dv = dx\, dy$$

$$I = \int_1^4 \int_1^3 e^{uv}\, \frac{u}{2}\, dv\, du. \qquad \boxed{\frac{\partial}{\partial v} e^{uv} = u e^{uv}}$$

$$= \int_1^4 \left(\frac{1}{2} e^{uv} \Big|_{v=1}^{v=3} \right) du$$

$$= \int_1^4 \frac{1}{2} \left(e^{3u} - e^{u} \right) du$$

$$= \frac{1}{2} \left(\frac{1}{3} e^{3u} - e^{u} \right) \Big|_1^4$$

$$= \frac{1}{2} \left(\frac{1}{3} e^{12} - e^4 - \frac{1}{3} e^3 + e \right)$$