MIDTERM 2, April 14 Friday
Beginning — 5.4.
First MT covered material up to half of 2.6

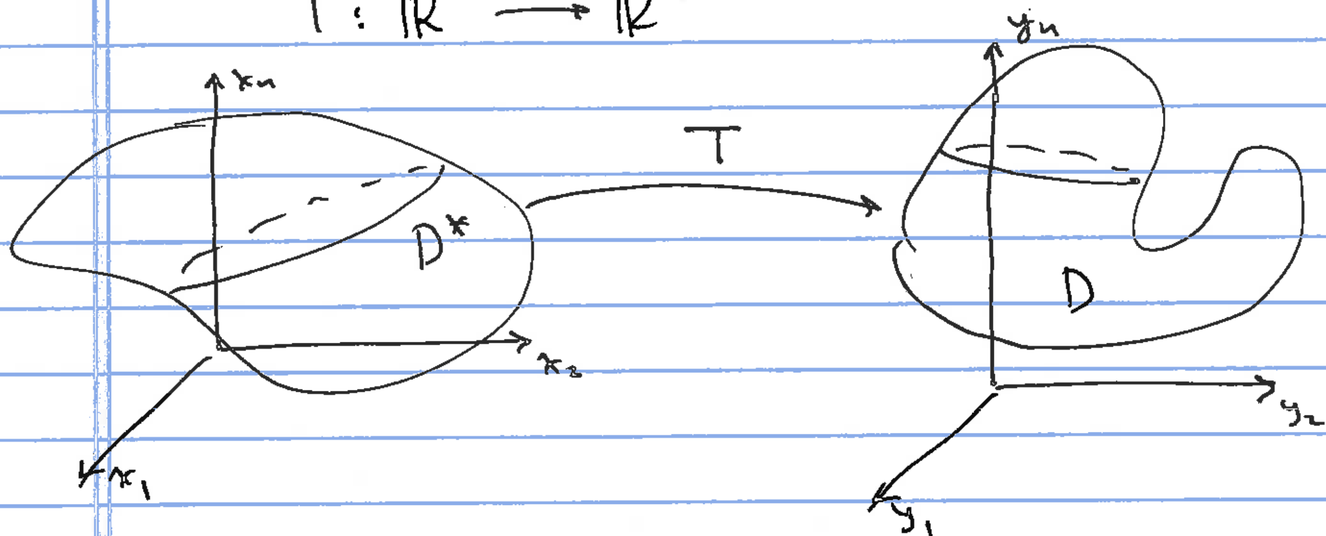
HW # 11 5.4 due Thursday April 13

Review Wed April 12 2hrs. (Room+ TBA
Poss. Time 6:30 - 8:30 ??

5.5 Change of Variables / substitution for integrals

I Coordinate transformations

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

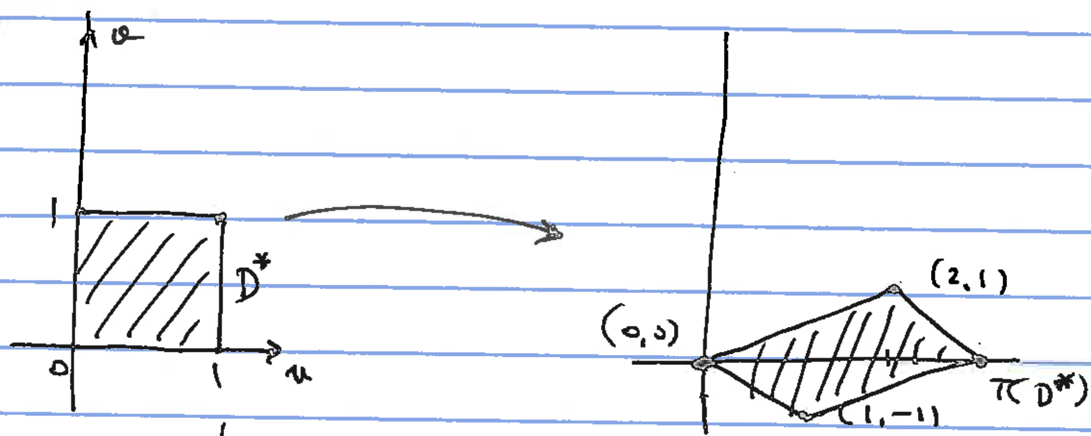


Want T to be 1-1, onto D

DT to be continuous C^1

$\det DT \neq 0$

Ex 1 $\quad T(u,v) = (2u+v,\ u-v) : \mathbb{R}^2 \to \mathbb{R}^2$



Linear maps take lines to lines.

check the vertices

$$\begin{cases} T(0,0) = (0,0) \\ T(0,1) = (1,-1) \\ T(1,0) = (2,1) \\ T(1,1) = (3,0) \end{cases}$$

$$DT = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \qquad \det DT = -3$$

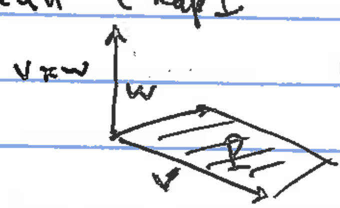
$$\Downarrow$$

$$|-3| \cdot \text{Area}(D^*) = \text{Area } T(D^*)$$

$$3 \cdot 1 = 3$$

compare

Recall Chap I



$|v \times w| = \text{area } P$

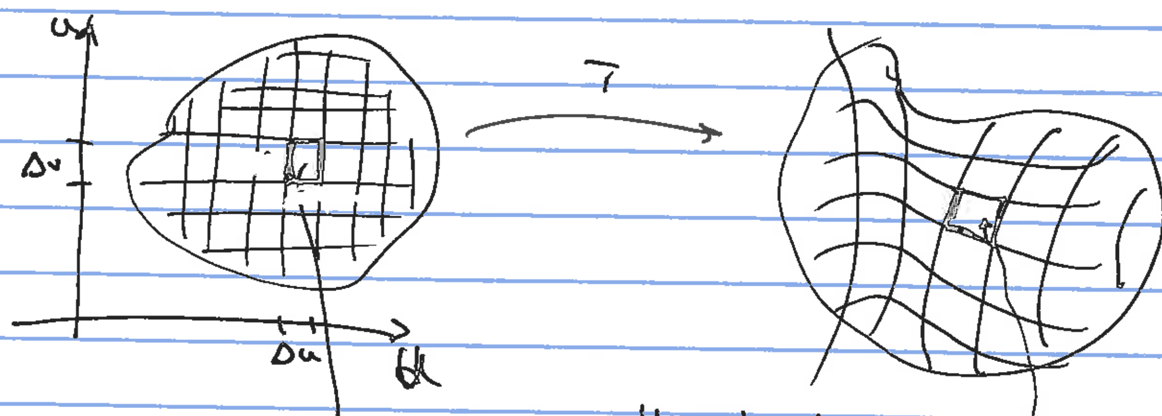
$$\begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \left(0, 0, \underbrace{\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}}_{-3}\right)$$

Thm: $L : \mathbb{R}^n \to \mathbb{R}^n$ linear map

$$\text{volume}_n(L(D^*)) = |\det DL| \cdot \text{volume}_n(D^*)$$

math notation
$$\begin{cases} \text{volume}_1 = \text{length} \\ \text{volume}_2 = \text{area} \\ \text{volume}_3 = \text{classical volume} \end{cases}$$

Main idea



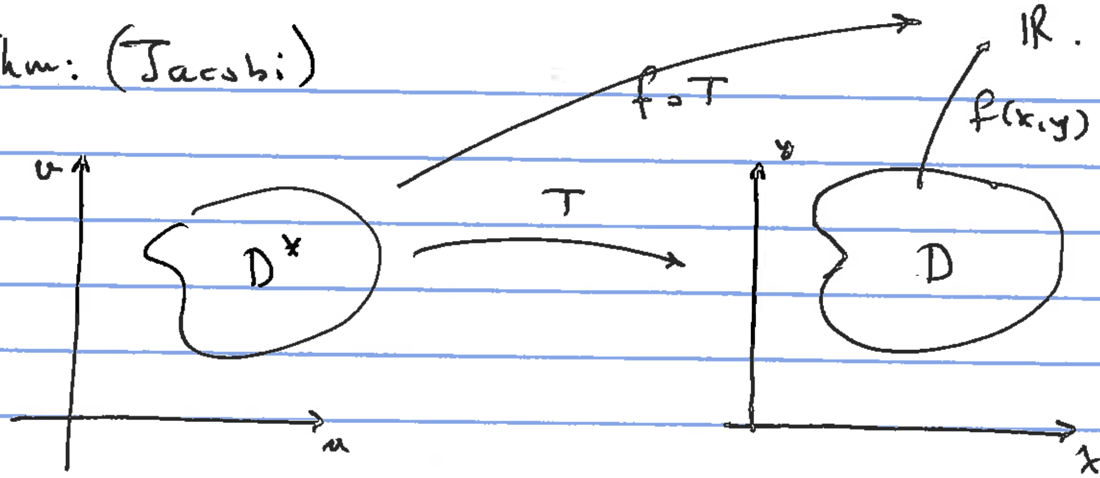
Area $= \Delta u \, \Delta v$ $\xrightarrow{\text{How does it change?}}$ Area of the twisted small almost rectangular region

??

$|\det DT| =$ factor of expansion/shrinkage of volume

Thm: (Jacobi)

n = 2



Let $T : D^* \longrightarrow D$ be a change of coordinates

T is 1-1, onto D

T is continuously diff'ble

$\det DT \neq 0$,

$f : D \longrightarrow \mathbb{R}$ be continuous.

Then: $$\iint_D f(x,y)\, \underset{dx\,dy}{\underbrace{dA}} = \iint_{D^* = T^{-1}(D)} f(T(u,v))\, |\det DT|\, \underset{du\,dv}{\underbrace{dA}}$$

Defn: $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \det DT$$

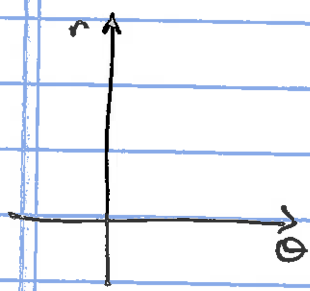
$(x,y) = T(u,v)$

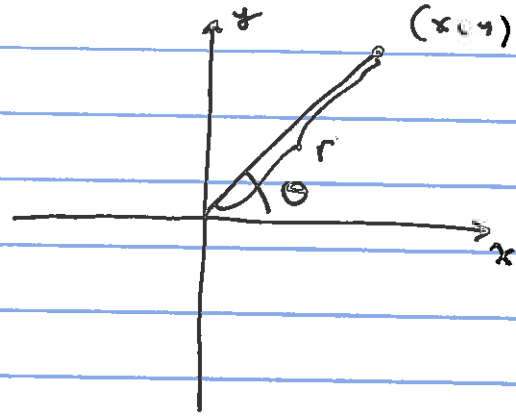
So: We will have

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = |\det DT|\, du\,dv$$

Polar Coordinates

Ex 1

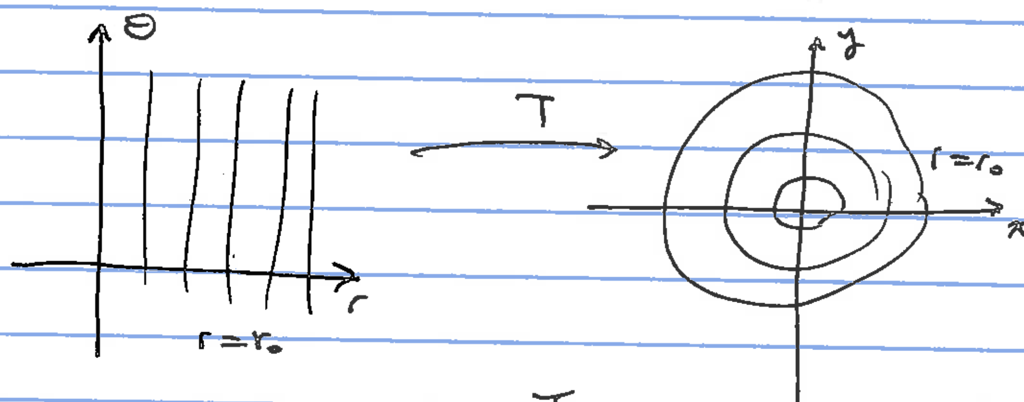


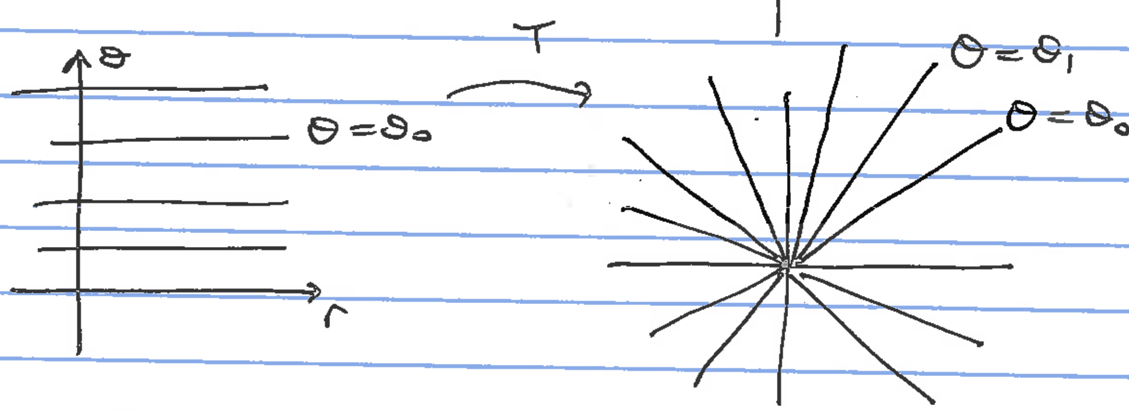


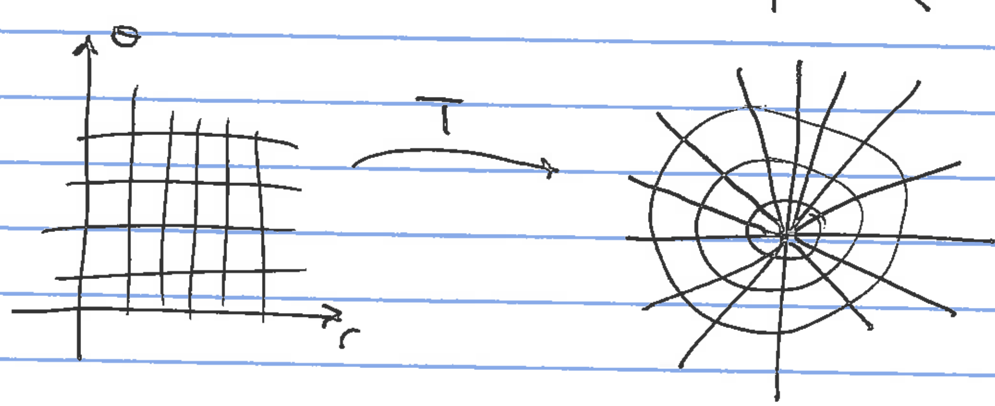
$x = r\cos\theta$

$y = r\sin\theta$

$$T(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y}) = (x,y)$$







$$DT = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

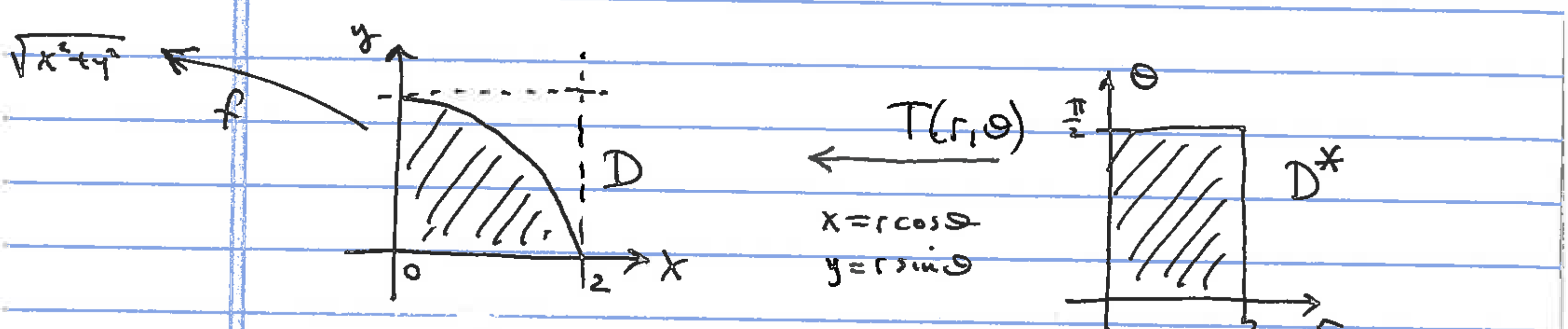
$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r.$$

$$dx\,dy = \underbrace{\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|}_{r} dr\,d\theta$$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$ for polar coordinates

Ex 1 (B) $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} \sqrt{x^2+y^2}\,dx\,dy$ Evaluate by using Polar Coordinates



$0 \le y \le 2$

$0 \le x \le \sqrt{4-y^2}$

$0 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$

$$T \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = r\,dr\,d\theta$$

$$\underbrace{\int_0^2 \int_0^{\sqrt{4-y^2}}}_{D} \sqrt{x^2+y^2}\,dx\,dy = \underbrace{\int_0^{\pi/2} \int_0^2}_{D^*} r \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^2 d\theta = \int_0^{\pi/2} \frac{8}{3}\,d\theta = \frac{8}{3}\cdot\frac{\pi}{2} = \frac{4\pi}{3}$$

<u>Next page: Compare to Calc I.</u>

Compare to Calculus I:

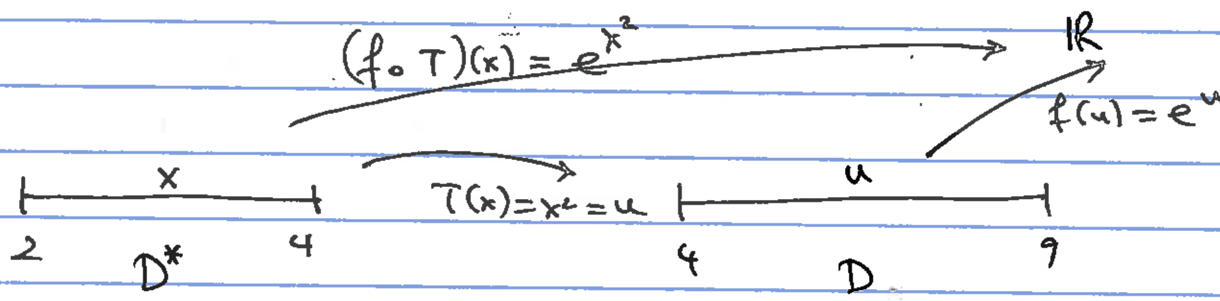
We want to Calculate $\int_2^4 2x e^{x^2} dx$.

Substitute $u = x^2$

$du = 2x\,dx$.

$$\int_2^4 2x e^{x^2} dx = \int_4^9 e^u du.$$

How does this relate to our new set up?



$$\int_2^4 e^{x^2} 2x\,dx = \int_{D^*} \underbrace{f(u(x))}_{e^{x^2}} \cdot \underbrace{\left|\frac{\partial u}{\partial x}\right|}_{|2x|} \underbrace{dx}_{dx} = \int_4^9 f(u)\,du = \int_4^9 e^u du$$