April 3, 2017

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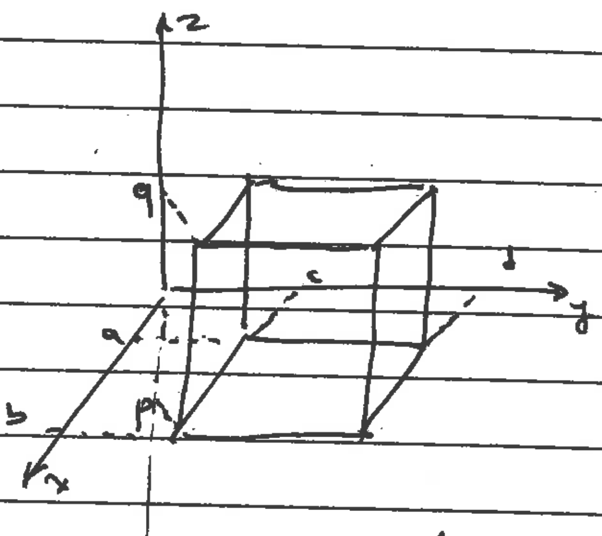
(5.4) Main idea of how to extend double to triple integrals

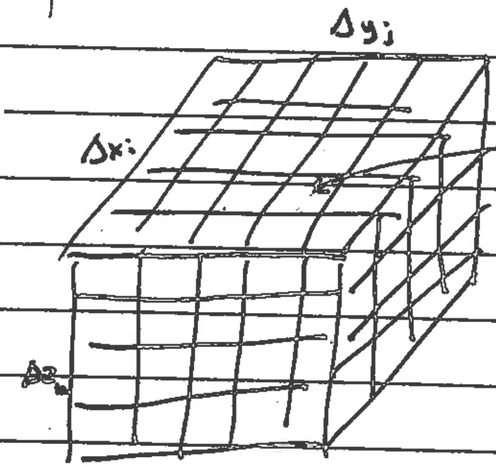
Let $f(x,y,z) = \underbrace{[a,b] \times [c,d] \times [p,q]}_{\text{Box } B} \longrightarrow \mathbb{R}$

$$\| $$

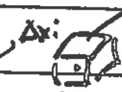
$$\left\{ (x,y,z) \;\middle|\; \begin{array}{c} a \le x \le b \\ c \le y \le d \\ p \le z \le q \end{array} \right\}$$




Slice parallel to coordinate planes




$f(c_{ijk})$




$c_{ijk} \in$ small box.

$$\mathcal{P} \doteq \begin{array}{l} a = x_0 < x_1 \ldots < x_n = b \\ c = y_0 < y_1 \ldots < y_m = d \\ p = z_0 < z_1 \ldots < z_\ell = q \end{array}$$

Partition

Riemann Sum

$$\sum_{i,j,k} \underbrace{f(c_{ijk})}_{\substack{\text{approximate}\\ \text{density}\\ \text{in small box}}} \cdot \underbrace{\Delta x_i \, \Delta y_j \, \Delta z_k}_{\substack{\text{volume}\\ \text{of small box}}} = R(f, \mathcal{P}, \{c_{ijk}\})$$

Approximation of the mass with density function $f$

$$\xrightarrow{\text{as } \|\mathcal{P}\| \to 0} \iiint_B f \, dV \quad (\text{if possible})$$

## FUBINI'S THM

Let $f : \underbrace{[a,b]\times[c,d]\times[p,q]}_{B} \longrightarrow \mathbb{R}$ be bounded

Let $S$ be the set of discontinuities of $f$.

If
- $S$ has $0$ volume, and
- All lines parallel to axes $x, y, z$ intersect $S$ at finitely many pts

then
$$\iiint_B f\,dV$$ exists, and equals to

$$\int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx$$

(as well as the other 5 orders.)

(There are 6 different orders)

Example #2 p 347.

$$\iiint\limits_{[0,1]\times[0,2]\times[0,3]} (x^2+y^2+z^2)\, dV.$$

By Fubini's Thm

$$= \int_0^1 \int_0^2 \int_0^3 (x^2+y^2+z^2)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^2 x^2 z + y^2 z + \frac{z^3}{3} \Big|_{z=0}^{z=3} dy\, dx$$

$$= \int_0^1 \int_0^2 (3x^2 + 3y^2 + 9) - 0 \; dy\, dx$$

$$= \int_0^1 3x^2 y + y^3 + 9y \Big|_{y=0}^{y=2} dx$$

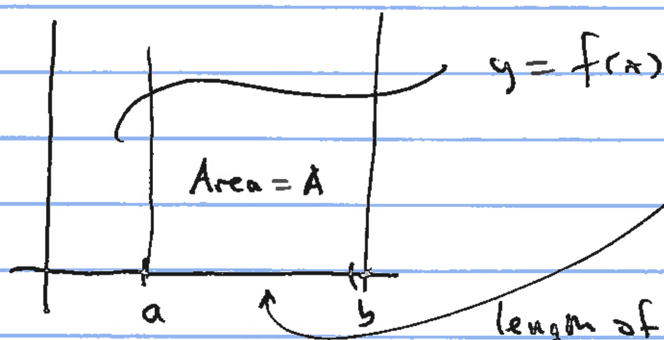
$$= \int_0^1 (6x^2 + 8 + 18 - 0)\, dx$$

$$= \int_0^1 (6x^2 + 26)\, dx$$

$$= 2x^3 + 26x \Big|_0^1 = (2 + 26) - 0 = 28$$

Compare, multiple integrals:

$n=1$.

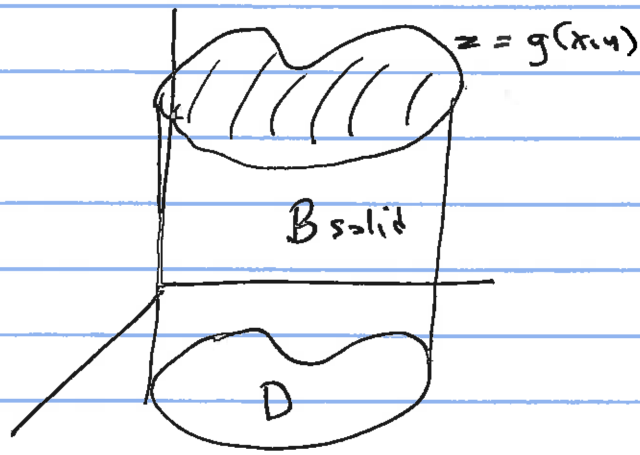



$$\int_a^b 1\,dx = b-a$$

$$\int_a^b f(x)\,dx = A$$

length of the domain

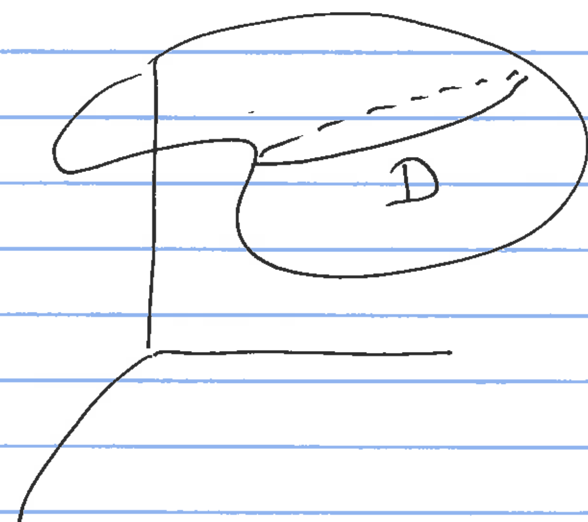
$n=2$




$$\iint_D 1\,dA = \text{area } D$$

$$\iint_D g(x,y)\,dA = \text{volume } B$$

$n=3$




$$\iiint_D 1\,dV = \text{volume } D$$

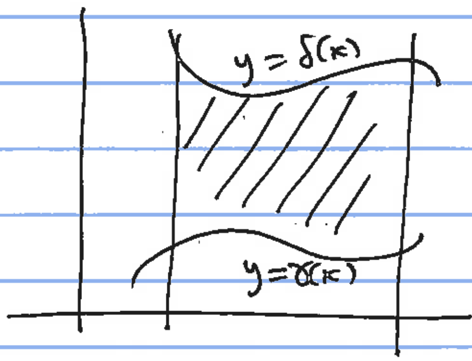
$$\iiint_D h(x,y,z)\,dV$$

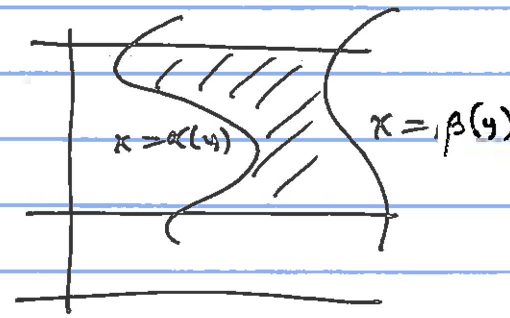
If $h(x,y,z)$ measurement at certain object/substance then $\iiint_V h\,dV$ will be the total amount of that substance.

# ELEMENTARY REGIONS

Recall

$n = 2$




$a \le x \le b$

$\gamma(x) \le y \le \delta(x)$

$c \le y \le d$

$\alpha(y) \le x \le \beta(y)$

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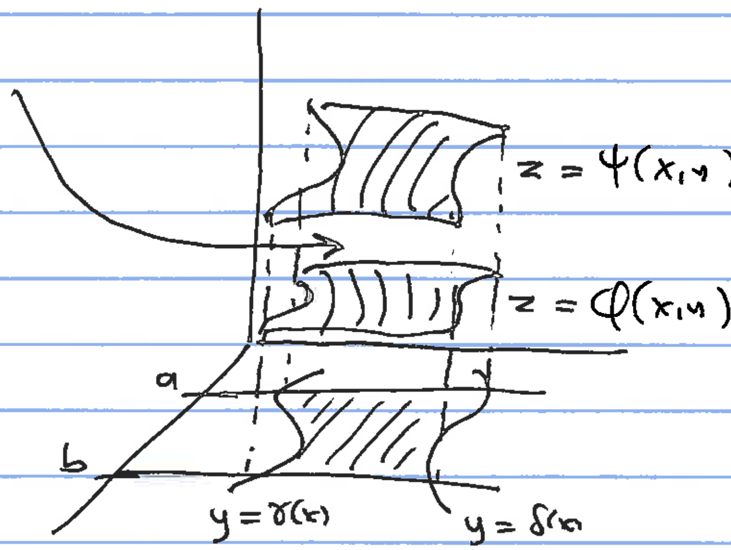
$n = 3$    6 Types of Elementary regions:

Type I:

$$\left.\begin{array}{c} a \le x \le b \\ \gamma(x) \le y \le \delta(x) \\ \varphi(x,y) \le z \le \psi(x,y) \end{array}\right\}$$




$$\int_a^b \int_{\gamma(x)}^{\delta(x)} \int_{\varphi(x,y)}^{\psi(x,y)} g(x,y,z) \, dz \, dy \, dx$$

Ex $$\int_0^2 \int_0^{2y} \int_{xy}^{3x+y} 2xy^2z \, dz \, dx \, dy$$

$$= \int_0^2 \int_0^{2y} xy^2z^2 \Big|_{z=xy}^{z=3x+y} dx \, dy$$

$$= \int_0^2 \int_0^{2y} xy^2\left[(3x+y)^2 - x^2y^2\right] dx \, dy$$

$$= \int_0^2 \int_0^{2y} xy^2\left[9x^2 + y^2 + 6xy - x^2y^2\right] dx \, dy$$

$$= \int_0^2 \int_0^{2y} \left(9x^3y^2 + xy^4 + 6x^2y^3 - x^3y^4\right) dx \, dy$$

$$= \int_0^2 \frac{9x^4y^2}{4} + \frac{x^2y^4}{2} + 2x^3y^3 - \frac{x^4y^4}{4} \Big|_{x=0}^{x=2y} dy$$

$$= \int_0^2 \frac{9(2y)^4 \cdot y^2}{4} + \frac{(2y)^2 \cdot y^4}{2} + \frac{2(2y)^3y^3}{1} - \frac{(2y)^4y^4}{4} dy$$

$$= \int_0^2 36y^6 + 2y^6 + 16y^6 - 4y^8 \, dy$$

$$= \int_0^2 54y^6 - 4y^8 \, dy = \frac{54y^7}{7} - \frac{4y^9}{9} \Big|_0^2$$

$$= \frac{54 \cdot 128}{7} - \frac{4 \cdot 512}{9} = \frac{47872}{63}$$

p348 #6

$$\int_1^3 \int_0^z \int_1^{xz} (x + 2y + z)\, dy\, dx\, dz$$

$$= \int_1^3 \int_0^z xy + y^2 + yz \Big|_{y=1}^{y=xz} dx\, dz$$

$$= \int_1^3 \int_0^z \left( x^2 z + x^2 z^2 + xz^2 - x - 1 - z \right) dx\, dz$$

$$= \int_1^3 \frac{x^3 z}{3} + \frac{x^3 z^2}{3} + \frac{x^2 z^2}{2} - \frac{x^2}{2} - x - xz \Big|_{x=0}^{x=z} dz$$

$$= \int_1^3 \left( \frac{z^4}{3} + \frac{z^5}{3} + \frac{z^4}{2} - \frac{z^2}{2} - z - z^2 \right) - 0 \; dz$$

$$= \left( \frac{z^5}{15} + \frac{z^6}{18} + \frac{z^5}{10} - \frac{z^3}{6} - \frac{z^2}{2} - \frac{z^3}{3} \right) \Big|_1^3$$

$$= \left( \frac{3^5}{15} + \frac{3^6}{18} + \frac{3^5}{10} - \frac{3^3}{6} - \frac{3^2}{2} - \frac{3^3}{3} \right) -$$

$$- \left( \frac{1}{15} + \frac{1}{18} + \frac{1}{10} - \frac{1}{6} - \frac{1}{2} - \frac{1}{3} \right)$$

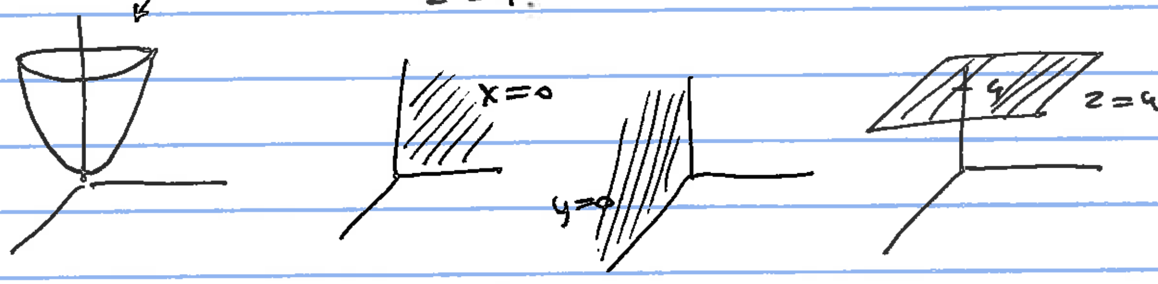
#16 p348 First octant $x,y,z \geq 0$

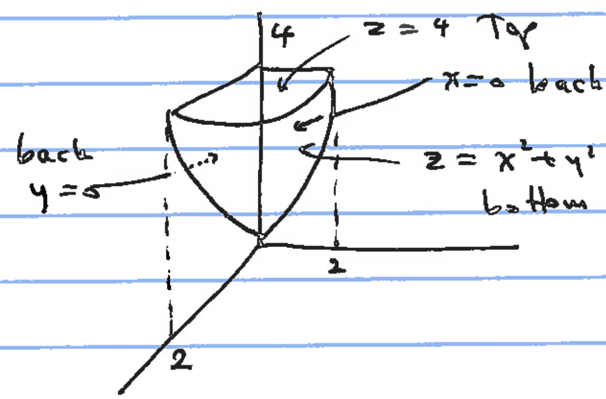
$z = x^2+y^2$

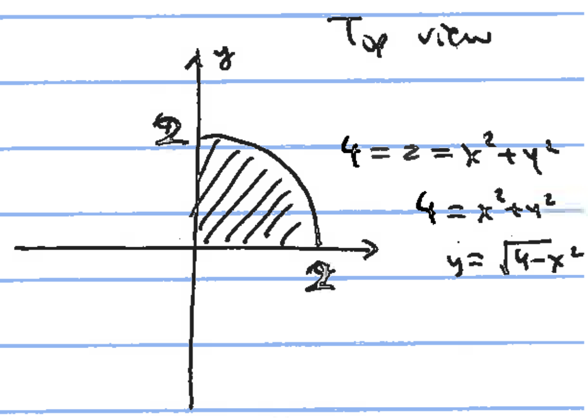
$x=0$

$y=0$

$z=4$.




$0 \leq x \leq 2$

$0 \leq y \leq \sqrt{4-x^2}$

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} 3y \; dz \; dy \; dx$$

To be continued.