March 31, 2017

1

5.3 Change the order of integration

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

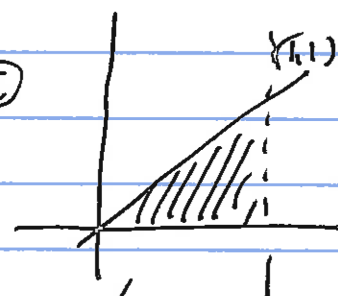
When all $a, b, c, d \in \mathbb{R}$.

5.3 #2 $\quad \mathrm{I} = \displaystyle\int_0^1 \int_0^x (2-x-y)\,dy\,dx$

$0 \le x \le 1$

$0 \le y \le x$

I



II

$0 \le y \le 1$

$y \le x \le 1$

$$\int_0^1 \int_y^1 (2-x-y)\,dx\,dy$$

Do either, neither is difficult.

I

$$\int_0^1 \int_0^x (2-x-y)\,dy\,dx = \int_0^1 (2-x)y - \frac{y^2}{2} \Big|_{y=0}^{y=x}$$

$$= \int_0^1 \left((2-x)x - \frac{x^2}{2} - 0\right) dx = \int_0^1 -\frac{3}{2}x^2 + 2x\,dx = -\frac{1}{2}x^3 + x^2 \Big|_0^1$$

$$= \frac{1}{2}.$$

Sec #14

Evaluate: $\displaystyle\int_0^1 \int_{3y}^3 \cos(x^2)\,dx\,dy$

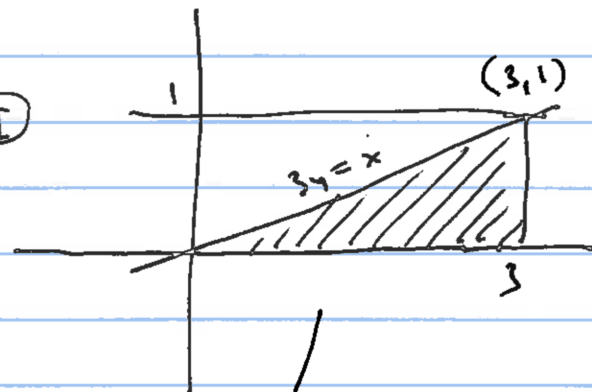
$\displaystyle\int \cos(x^2)\,dx$ is not expressible in terms of finitely many simple functions.

Try to reverse the order of integration

$0 \le y \le 1$

$3y \le x \le 3$

I



$3y = x$

$y = \frac{x}{3}$

II

$0 \le x \le 3$

$0 \le y \le \frac{x}{3}$

$$\int_0^3 \int_0^{x/3} \cos(x^2)\,dy\,dx = \int_0^3 y\cos x^2 \Big|_0^{x/3} dx$$

$$= \int_0^3 \frac{x}{3}\cos x^2\,dx = \int_0^9 \frac{1}{6}\cos u\,du =$$

$u = x^2$

$du = 2x\,dx \longrightarrow \frac{1}{6}du = \frac{x}{3}dx$

$$= \frac{1}{6}\sin u\Big|_0^9 = \frac{1}{6}\sin 9$$

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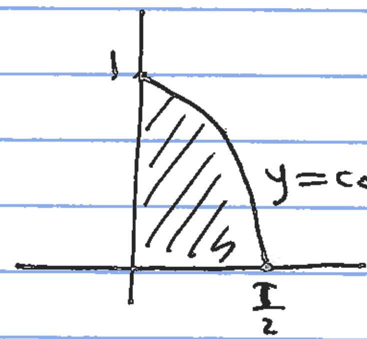
I $\displaystyle\int_0^{\pi/2}\int_0^{\cos x} \sin x \, dy\, dx$

$0 \le x \le \pi/2$

$0 \le y \le \cos x$



$y = \cos x$

$\cos^{-1} y = x$

$0 \le y \le 1$

$0 \le x \le \cos^{-1} y$

II $\displaystyle\int_0^{1}\int_0^{\cos^{-1} y} \sin x \, dx\, dy$ → next page

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I $\displaystyle\int_0^{\pi/2}\int_0^{\cos x} \sin x \, dy\, dx = \int_0^{\pi/2} y \sin x \Big|_{y=0}^{y=\cos x} dx$

$\displaystyle = \int_0^{\pi/2} \underbrace{\cos x \sin x}_{\frac{1}{2}\sin 2x} \, dx = \int_0^{\pi/2} \frac{1}{2}\sin 2x \, dx$

$\displaystyle = -\frac{1}{4}\cos 2x \Big|_0^{\pi/2}$

$\displaystyle = -\frac{1}{4}\Big(\underbrace{\cos \pi}_{-1} - \underbrace{\cos 0}_{+1}\Big) = \frac{1}{2}.$

OR

$u = \sin x$

$du = \cos x \, dx$ $\displaystyle \quad I = \int_0^1 u\, du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

II $\displaystyle\int_0^1 \int_0^{\cos^{-1} y} \sin x \, dx \, dy$

$$= \int_0^1 -\cos x \Big|_{x=0}^{x=\cos^{-1} y} dy$$

$$= \int_0^1 -\Big[\cos(\cos^{-1} y) - \underbrace{\cos 0}_{1}\Big] dy$$

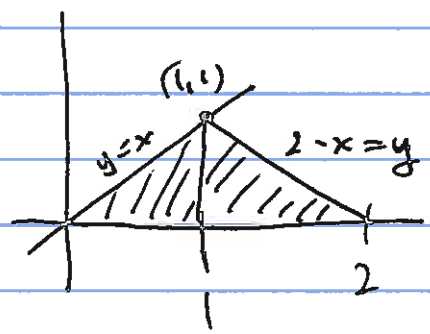
$$= \int_0^1 (1-y)\,dy = \left(y - \frac{y^2}{2} \Big|_0^1 \right) = 1 - \frac{1}{2} = \frac{1}{2}$$

Exc # 12

$$\int_0^1 \int_0^x \sin x \, dy\, dx + \int_1^2 \int_0^{2-x} \sin x \, dy\, dx$$

$0 \le x \le 1$ $\qquad$ $1 \le x \le 2$

$0 \le y \le x$ $\qquad$ $0 \le y \le 2-x$



$0 \le y \le 1$

$y \le x \le 2-y$

$x+y=2$

$x = 2-y$

$$\int_0^1 \int_y^{2-y} \sin x \, dx\, dy$$

$$= \int_0^1 -\cos x \Big|_{x=y}^{x=2-y} dy$$

$$= \int_0^1 -\cos(2-y) + \cos y \, dy$$

$$= \sin(2-y) + \sin y \Big|_0^1$$

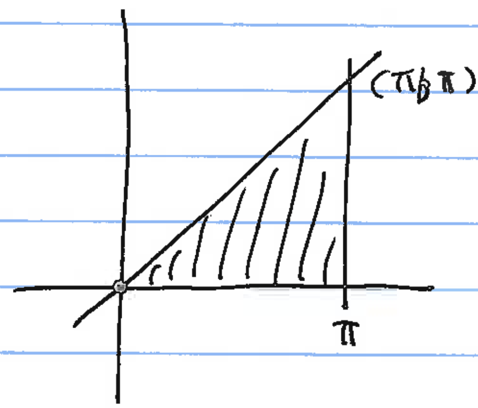
$$= (\sin 1 + \sin 1) - (\sin 2 + \cancel{\sin 0})$$

$$= 2\sin 1 - \sin 2$$

5.3 #16 $\displaystyle\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\, dx\, dy$ Evaluate

II $0 \le y \le \pi$

$y \le x \le \pi$



$\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\Rightarrow$ f is continuously extendible to (0, 0).

$\Rightarrow$ f is integrable

$\Rightarrow$ change of order is allowed

I $0 \le x \le \pi$

$0 \le y \le x$

$$\int_0^{\pi}\int_0^{x} \frac{\sin x}{x}\, dy\, dx = \int_0^{\pi} \frac{\sin x}{x} \cdot y \Big|_{y=0}^{y=x} dx$$

$$= \int_0^{\pi} \left(\frac{\sin x}{\cancel{x}} \cdot \cancel{x} - 0 \right) dx$$

$$= \int_0^{\pi} \sin x\, dx = -\cos x \Big|_0^{\pi} = -\underbrace{\cos \pi}_{-1} + \cos 0$$

$$= 2.$$