To Finish 5.1

Exc #8 5.1

March 27, 2017

1

Ex

$$\int_1^2 \int_0^3 (x+3y+1)\,dx\,dy$$

$$= \int_1^2 \left(\frac{x^2}{2} + 3xy + x \Big|_{x=0}^{x=3} \right) dy$$

$$= \int_1^2 \left(\frac{9}{2} + 9y + 3\right) - (0)\,dy$$

$$= \int_1^2 \frac{15}{2} + 9y\,dy$$

$$= \frac{15}{2}y + \frac{9}{2}y^2 \Big|_{y=1}^{y=2}$$

$$= (15 + 18) - \underbrace{\left(\frac{15}{2} + \frac{9}{2}\right)}_{12} = 21.$$

We Try the order order go to 1b

1b

$$\int_0^3 \int_1^2 (x+3y+1)\,dy\,dx$$

$$= \int_0^3 \left(xy + \frac{3y^2}{2} + y \Big|_{y=1}^{y=2} \right) dx$$

$$= \int_0^3 \left((2x + 6 + 2) - \left(x + \frac{3}{2} + 1\right) \right) dx$$

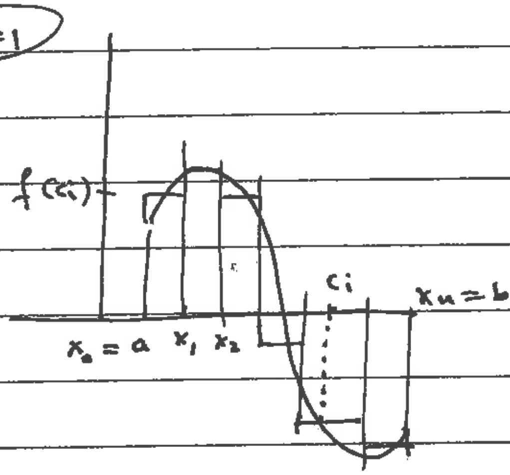
$$= \int_0^3 \left(x + \frac{11}{2} \right) dx$$

$$= \frac{x^2}{2} + \frac{11}{2}x \Big|_{x=0}^{x=3}$$

$$= \frac{9}{2} + \frac{33}{2} = 21.$$

Review of Calc I, Riemann Integrals

$n=1$



Partition P : $a = x_0 < x_1 \cdots < x_n = b$

$c_i \in [x_{i-1}, x_i]$

$\Delta x_i = x_i - x_{i-1}$

$\|P\| = \max \Delta_i$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = R(f, P, \{c_i\}) \xrightarrow[\|P\| \to 0]{} \int_a^b f(x)\,dx$$
as $\|P\| \to 0$

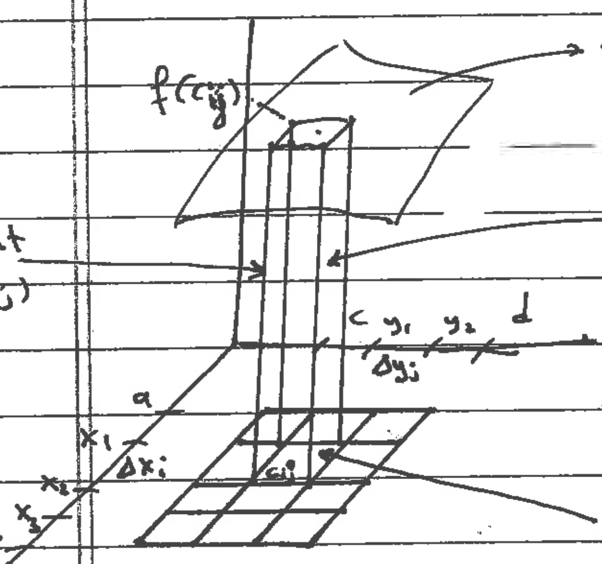
5.2 $n=2$

R (rectangle)

Caution ($R \neq \mathbb{R}$)

Def let $f : [a,b] \times [c,d] \to \mathbb{R}$ be bounded.

$z = f(x,y)$



volume $= f(c_{ij}) \cdot \Delta x_i \Delta y_j$

height $f(c_{ij})$

Let $a = x_0 < x_1 \cdots < x_k = b$, $c = y_0 < y_1 \cdots < y_\ell = d$ $\Big\} P$

Area $= \Delta A_{ij} = \Delta x_i \Delta y_j = |x_i - x_{i-1}||y_j - y_{j-1}|$

$\|P\| = \max(\Delta x_i, \Delta y_j)$

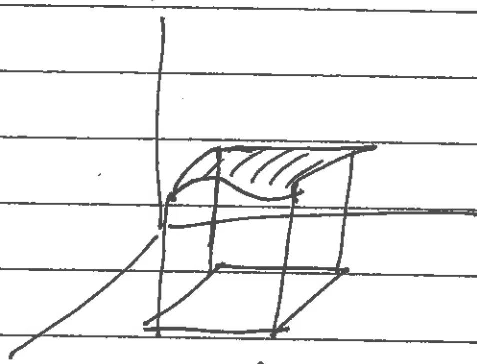
Let $c_{ij} \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

If $$R(f, P, \{c_{ij}\}) = \sum_{\substack{i=1\\ j=1}}^{k,\ell} f(c_{ij})\Delta x_i \Delta y_j \xrightarrow[\text{as } \|P\| \to 0]{\text{limit}} L \in \mathbb{R};$$

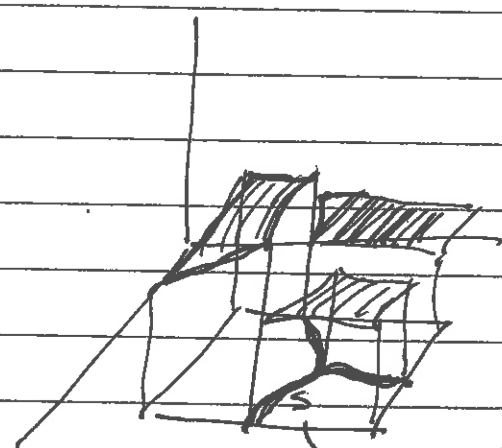
then f is called integrable and $L = \iint_R f\,dA$.

5.2

Thm: $f: [a,b] \times [c,d] \to \mathbb{R}$ is integrable if f is continuous (OR the set S of discontinuities of f has 0 area)



$S = \phi$



The set of discontinuities

FUBINI'S THEOREM

Let (i) $f: [a,b] \times [c,d] \to \mathbb{R}$ be bounded

(ii) $S = \{(x,y) \mid f \text{ is not continuous}\}$, the set of discontinuities have 0 area.

(iii) S intersect lines parallel to x or y-axes at finitely many points.

Then (i) f is integrable, and

(ii) $$\iint_R f\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Example (5.2)

Calculate the double integral of $f(x,y) = (16 - x^2 - y^2)$ over the rectangle $[-2,2] \times [1,3]$.

$$\iint_{[-2,2]\times[1,3]} (16 - x^2 - y^2)\, dA \overset{\mathrm{I}}{=} \int_{-2}^{2} \int_{1}^{3} (16 - x^2 - y^2)\, dy\, dx$$

$$= \overset{\mathrm{II}}{=} \int_{1}^{3} \int_{-2}^{2} (16 - x^2 - y^2)\, dx\, dy$$

Fubini's Thm.

I $\displaystyle\int_{-2}^{2} \int_{1}^{3} (16 - x^2 - y^2)\, dy\, dx$

$$= \int_{-2}^{2} 16y - x^2 y - \frac{y^3}{3} \Big|_{y=1}^{y=3} dx.$$

$$= \int_{-2}^{2} \Big(\big(\underbrace{48 - 3x^2 - 9}_{39\ -16+\frac{1}{3}} \big) - \big(16 - x^2 - \tfrac{1}{3} \big) \Big) dx$$

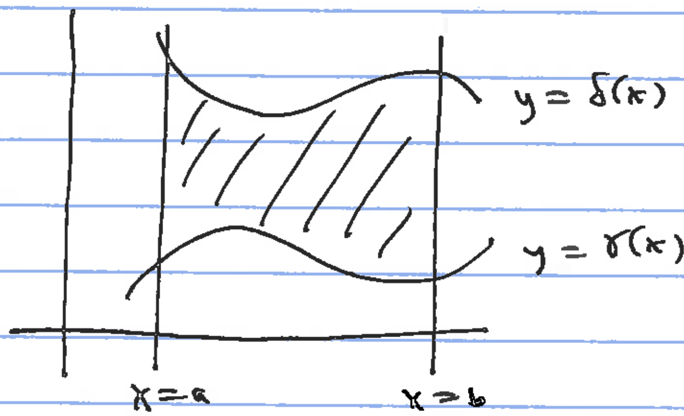
$$= \int_{-2}^{2} \left(-2x^2 + 23\tfrac{1}{3} \right) dx = -\frac{2}{3}x^3 + \frac{70}{3}x \Big|_{x=-2}^{x=2}$$

$$= \left(-\frac{16}{3} + \frac{140}{3}\right) - \left(\frac{16}{3} - \frac{140}{3}\right)$$

$$= 2\left(\frac{140}{3} - \frac{16}{3}\right) = 2\left(\frac{124}{3}\right) = \frac{248}{3},$$

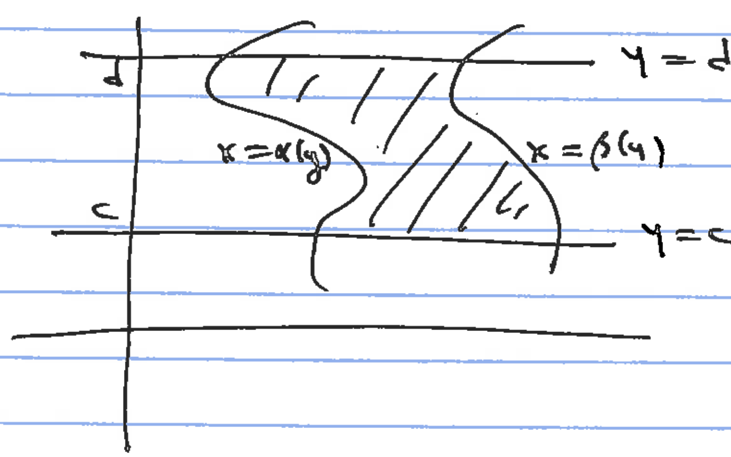
ELEMENTARY REGIONS

Type I



$$\{(x,y) \mid a \leq x \leq b \text{ and } \gamma(x) \leq y \leq \delta(x)\}$$

Type II



$$\{(x,y) \mid c \leq y \leq d \quad \alpha(y) \leq x \leq \beta(y)$$

Theorem I Let $f : D \longrightarrow \mathbb{R}$ be continuous & D be of type I

$$D = \{(x,y) \mid a \le x \le b, \ \gamma(x) \le y \le \delta(x)\}$$

Then

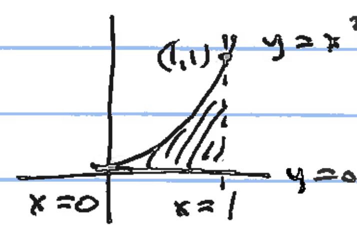
$$\iint_D f \, dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y) \, dy \, dx$$

Exc p 332 #4

$$\int_0^1 \int_0^{x^2} 3 \, dy \, dx$$

$0 \le x \le 1$

$0 \le y \le x^2$



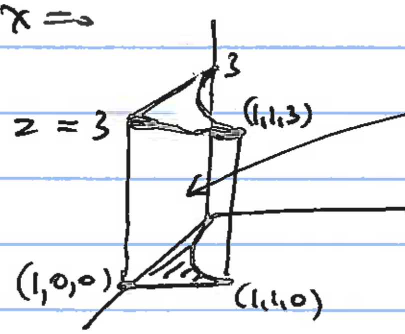
Domain of integration

$$= \int_0^1 3y \Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^1 (3x^2 - 0) \, dx$$

$$= x^3 \Big|_{x=0}^{x=1} = 1^3 - 0^3 = 1$$

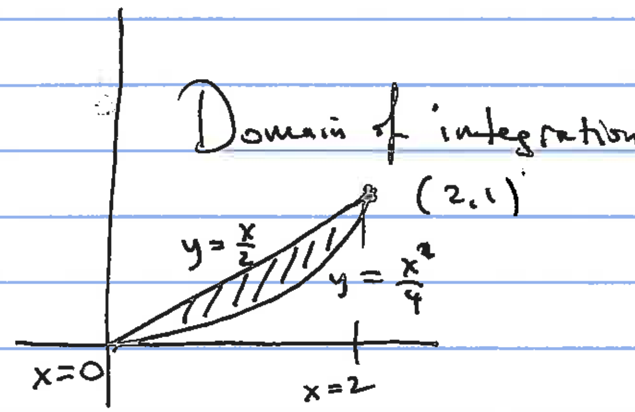
volume .



p332 #8 $\displaystyle\int_0^2 \int_{x^2/4}^{x/2} (x^2+y^2)\,dy\,dx.$

Sketch domain of integration

$$0 \le x \le 2$$
$$\frac{x^2}{4} \le y \le \frac{x}{2}$$



$$I = \int_0^2 \int_{x^2/4}^{x/2} (x^2+y^2)\,dy\,dx$$

$$= \int_0^2 x^2 y + \frac{y^3}{3}\Big]_{y=\frac{x^2}{4}}^{y=\frac{x}{2}} dx$$

$$= \int_0^2 \left(\left(x^2\cdot\frac{x}{2}\right) + \left(\frac{x}{2}\right)^3\cdot\frac{1}{3}\right) - \left(\left(x^2\cdot\frac{x^2}{4}\right) + \frac{1}{3}\left(\frac{x^2}{4}\right)^3\right.$$

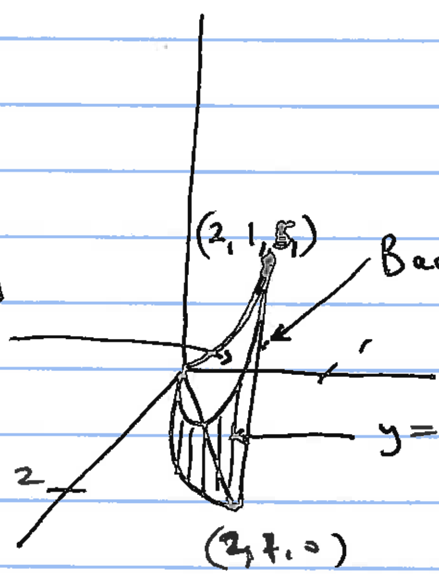
$$= \int_0^2 \left(\frac{x^3}{2} + \frac{x^3}{24} - \frac{x^4}{4} - \frac{x^6}{192}\right) dx$$

$$= \frac{x^4}{8} + \frac{x^4}{96} - \frac{x^5}{20} - \frac{x^7}{7\cdot 192}\Big|_0^2$$

$$= \frac{16}{8} + \frac{16}{96} - \frac{32}{20} + \frac{\overset{2}{\cancel{128}}}{7 \cdot \underset{3}{\cancel{192}}}$$

$$= 2 + \frac{1}{6} - \frac{8}{5} - \frac{2}{21} = \frac{33}{70}$$

Top is paraboloid $z = x^2 + y^2$



Back side $y = \frac{x}{2}$ plane

$y = \frac{x^2}{4}$ cylinder / side