4.3 Lagrange Multipliers Examples

LinEaR Algebra trick.

$$\underset{n \times n}{A} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (*)$$

$\det A \neq 0$, A is invertible $\Longrightarrow$ $\vec{x} = \vec{0}$ is the only solution.

If $\exists\, \vec{x} \neq \vec{0}$ solution of $(*)$ $\Longrightarrow$ $\det A = 0$.

Ex 4 Max/min $f(x,y) = (x-2y)^2$ subject to $x^2+y^2=1$.

$$\nabla f = \left(2(x-2y), -4(x-2y) \right)$$

$$\nabla g = (2x, 2y)$$

$$\nabla f = \lambda \nabla g$$

$$2(x-2y) = 2\lambda x \quad = 2x - 4y$$
$$-4(x-2y) = 2\lambda y \quad = -4x + 8y$$

$$x^2+y^2 = 1.$$

① $2\lambda x - 2x + 4y = 0$ +   ③ $x^2 + y^2 = 1$

② $4x + 2\lambda y - 8y = 0$

$$\underbrace{\begin{matrix} x(2\lambda - 2) + 4y = 0 \\ 4x + (2\lambda - 8)y = 0 \end{matrix}}_{\text{want non-zero sol}^n} \quad + \quad x^2 + y^2 = 1$$

$x = 0 = y$ is not a soln of

need

$$\begin{vmatrix} 2\lambda - 2 & 4 \\ 4 & 2\lambda - 8 \end{vmatrix} = 0$$

$$(2\lambda - 2)(2\lambda - 8) - 16 = 0$$

$$4\lambda^2 - 16\lambda - 4\lambda + \cancel{16} - \cancel{16} = 0$$

$$4\lambda^2 - 20\lambda = 0$$

$$4\lambda(\lambda - 5) = 0$$

$\lambda = 0$ OR $\lambda = 5$

Case 1 $\lambda = 0$

① $-2x + 4y = 0$

② $4x - 8y = 0$

multiples of each other

wanted non-zero sol^n.

$$2x = 4y$$

$$x = 2y$$

$$1 = x^2 + y^2 = (2y)^2 + y^2 = 5y^2$$

$$y = \pm \frac{1}{\sqrt{5}} \qquad x = \pm \frac{2}{\sqrt{5}}$$

$$\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$$

Case 2 $\lambda = 5$ ① $10x - 2x + 4y = 0$

② $4x + 10y - 8y = 0$

$$\left.\begin{array}{l} 8x + 4y = 0 \\ 4x + 2y = 0 \end{array}\right\rangle \text{ multiples}$$

$$y = -2x.$$

$$1 = x^2 + y^2 = x^2 + (-2x)^2 = 5x^2$$

$$x = \pm \frac{1}{\sqrt{5}}, \quad y = \mp \frac{2}{\sqrt{5}}$$

$$\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right), \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

	$f = (x-2y)^2$
$\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$	0
$\left(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$	0
$\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$	$\left(\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}}\right)^2 = 5$
$\left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$	$\left(-\frac{1}{\sqrt{5}} - \frac{4}{\sqrt{5}}\right)^2 = 5$

Constraint set $x^2 + y^2 = 1$ compact $\Longrightarrow$ (max / min exist) ExtV Th

Max 5 at $\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right), \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

min 0 at $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right), \left(\frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$

Ex 5 Multiple Constraints

max/min $f = \frac{x^2+y^2}{2}$

subject to $\begin{cases} g_1 = x^2+y^2+z^2 = 1 \\ g_2 = x+y+z = 0 \end{cases}$ and

$$\nabla f = (x, y, 0)$$

$$\nabla g_1 = (2x, 2y, 2z)$$

$$\nabla g_2 = (1, 1, 1)$$

LM: $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$$(x, y, 0) = \lambda(2x, 2y, 2z) + \mu(1,1,1)$$

$$\left.\begin{array}{ll} ① & x = 2\lambda x + \mu \\ ② & y = 2\lambda y + \mu \\ ③ & 0 = 2\lambda z + \mu \\ & \\ ④ & x^2+y^2+z^2 = 1 \\ ⑤ & x+y+z = 0 \end{array}\right\} \text{solve}$$

1 − 2 $\qquad x - y = (2\lambda x + \mu) - (2\lambda y + \mu)$

$$x - y = 2\lambda(x - y)$$

$$0 = 2\lambda(x-y) - (x-y)$$

$$0 = (x-y)(2\lambda - 1)$$

OR

$x - y = 0$

$x = y$

4 $\Rightarrow 2x^2 + z^2 = 1$

5 $\Rightarrow 2x + z = 0$

$z = -2x$

4 $\Rightarrow 2x^2 + (-2x)^2 = 1$

$6x^2 = 1$

$x = \pm \frac{1}{\sqrt{6}} = y$

$z = \mp \frac{2}{\sqrt{6}}$

This is consistent w/

$\lambda = 1/6$

$\mu = \pm \frac{\sqrt{6}}{9}$

$2\lambda - 1 = 0$

$\lambda = \frac{1}{2}$

1 $\quad x = x + \mu$

2 $\quad y = y + \mu$

3 $\quad 0 = z + \mu$

4 $\quad x^2 + y^2 + z^2 = 1$

5 $\quad x + y + z = 0$

1 $\Rightarrow \mu = 0.$

3 $\Rightarrow z = 0$

4 $\Rightarrow x^2 + y^2 = 1$

5 $\Rightarrow x + y = 0$

$x = -y$

$x^2 + (-x)^2 = 1$

$2x^2 = 1$

$x = \pm \frac{1}{\sqrt{2}}$

$y = \mp \frac{1}{\sqrt{2}}$

$z = 0$

	$f = \frac{1}{2}(x^2+y^2)$	
$\left(\frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{-2}{\sqrt6}\right)$	$\frac{1}{2}\left(\frac{1}{6}+\frac{1}{6}\right) = \frac{1}{6}$	min
$\left(\frac{-1}{\sqrt6}, \frac{-1}{\sqrt6}, \frac{2}{\sqrt6}\right)$	$\frac{1}{6}$	
$\left(\frac{1}{\sqrt2}, \frac{-1}{\sqrt2}, 0\right)$	$\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right)$ $\frac{1}{2}$	max
$\left(\frac{-1}{\sqrt2}, \frac{1}{\sqrt2}, 0\right)$	$\frac{1}{2}$	

$x^2+y^2+z^2=1$

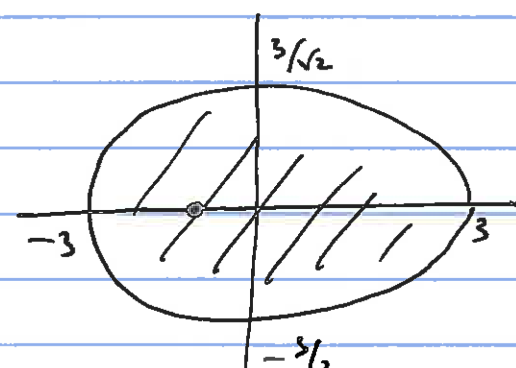
$x+y+z=0$

Compact

Ex 6 max/min $x^2+2x+4y^2 = f$

subject to $x^2+2y^2 \le 9$.

$$\frac{x^2}{9}+\frac{2}{9}y^2 \le 1$$



I Use 4.2 for interior $x^2+2y^2<9$

II Use 4.3 LM for boundary $x^2+2y^2=9$

I $\nabla f = (2x+2,\ 8y)$

Ist DT. $\nabla f = 0$

$$2x+2=0 \qquad x=-1$$
$$8y=0 \qquad y=0$$

only interior cp.

$H_f = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \Rightarrow (-1,0)$ is a local min.

Boundary $g = x^2 + 2y^2 = 9$

I LM $\nabla f = (2x+2, 8y)$

$\nabla g = (2x, 4y)$

$\nabla f = \lambda \nabla g$

$(2x+2, 8y) = \lambda(2x, 4y)$

$$\left.\begin{array}{ll} ① & 2x+2 = 2\lambda x \\ ② & 8y = 4\lambda y \\ ③ & x^2+2y^2 = 9 \end{array}\right\} \text{ solve}$$

② $8y - 4\lambda y = 0 = 4y(2-\lambda)$

$y = 0$ OR $\lambda = 2$

③ $x^2 = 9$

$x = \pm 3$

$(3, 0)$

$(-3, 0)$

① $2x + 2 = 4x$

$2 = 2x$

$1 = x$

$x^2 + 2y^2 = 9$

$1 + 2y^2 = 9$

$2y^2 = 8$

$y^2 = 4$

$y = \pm 2$

	c.p.	$x^2 + 2x + 4y^2$	
interior	$(-1, 0)$	-1	← min value
boundary	$(3, 0)$	15	
	$(-3, 0)$	3	
	$(1, 2)$	19	max value
	$(1, -2)$	19