March 22, 2017

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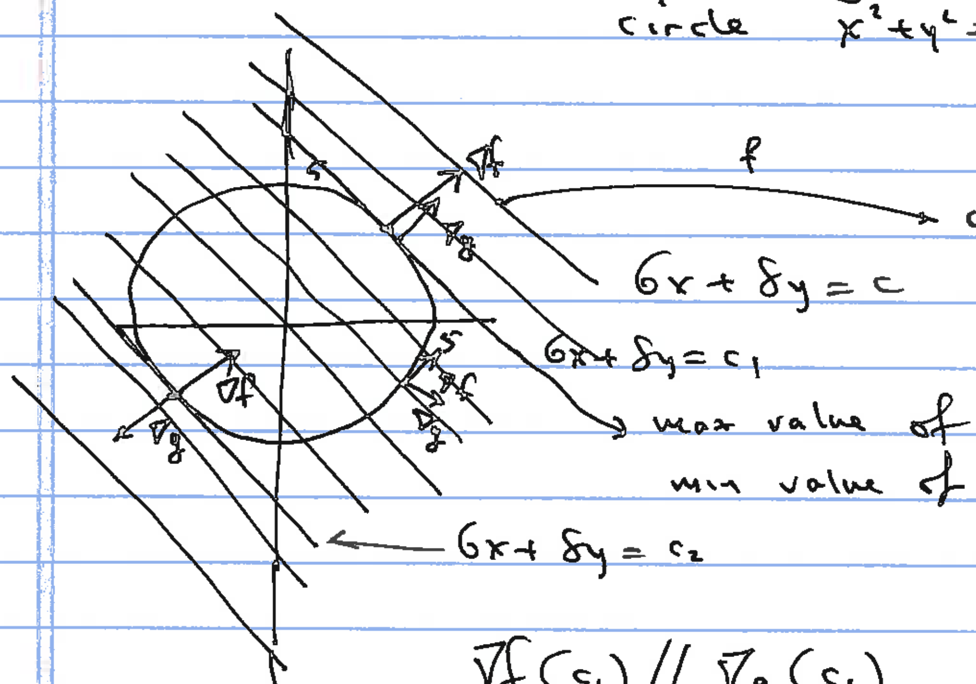
(4.3) Lagrange Multipliers

Ex 1 Max/min $f = 6x + 8y$ on the circle $x^2+y^2=25$



max value of f on circle is c_1

min value of f on circle is c_2

$$\nabla f(c_1) \parallel \nabla g(c_1)$$
$$\nabla f(c_2) \parallel \nabla g(c_2)$$

max/min $f = 6x+8y$ $\qquad$ $g = x^2+y^2 = 25$ constraint

$\nabla f = (6,8)$ $\qquad$ $\nabla g = (2x, 2y)$

$$(6,8) \parallel (2x,2y)$$

$$(6,8) = \lambda(2x,2y) \quad \text{for some } \lambda.$$

$$\left.\begin{array}{ll} ① & 6 = 2\lambda x \\ ② & 8 = 2\lambda y \\ ③ & x^2+y^2 = 25 \end{array}\right\} \text{solve}$$

Ex 1 Continue

① $\Rightarrow$ $x = \dfrac{6}{2\lambda} = \dfrac{3}{\lambda}$. ($\lambda \neq 0$ since $2\lambda x = 6$)

② $\Rightarrow$ $y = \dfrac{8}{2\lambda} = \dfrac{4}{\lambda}$

③ $x^2 + y^2 = 25 = \left(\dfrac{3}{\lambda}\right)^2 + \left(\dfrac{4}{\lambda}\right)^2 = \dfrac{9}{\lambda^2} + \dfrac{16}{\lambda^2} = \dfrac{25}{\lambda^2}$

$$25 = \frac{25}{\lambda^2}$$

$$\lambda^2 = 1.$$

$$\lambda = \pm 1$$

$\lambda = 1$ $x = 3$, $y = 4$ $\lambda = -1$, $x = -3$, $y = -4$.

	$6x + 8y$	
$(3,4)$	$18 + 32 = 50$	max
$(-3,-4)$	$-18 - 32 = -50$	min

$\{(x,y) \mid x^2 + y^2 = 25\}$ compact

$\Downarrow$ f cont

$\exists$ max/min

Max value is 50

min value is -50.

THEOREM (Lagrange Multipliers)

Let $f, g : \underset{\text{open}}{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ be both continuously diff'ble.

If $x_0 \in S = \{ \vec{x} \in X \mid g(x) = c \}$ and

if x_0 is a local extremum of

$f|S$ (f restricted to S), then $\exists \lambda \in \mathbb{R}$

s.t. $\nabla f(x_0) = \lambda \nabla g(x_0)$.

Ex 2 max/min $f = x + y^2$ with constraint $g = 4x^2 + y^2 = 1$.

$\nabla f = (1, 2y)$
$\nabla g = (8x, 2y)$

$$(1, 2y) = \lambda (8x, 2y)$$

$$\left.\begin{aligned} &① \quad 1 = 8\lambda x \\ &② \quad 2y = 2\lambda y \\ &③ \quad 4x^2 + y^2 = 1 \end{aligned}\right\} \quad \underline{\text{solve}}$$

Ex2 continue

② $\Longrightarrow$ $2y - 2\lambda y = 0$

$$2y(1-\lambda) = 0$$

OR

$y = 0$ | $\lambda = 1.$

③ $\Rightarrow$ $4x^2 + y^2 = 1$

$$4x^2 = 1$$

$$x^2 = \frac{1}{4}$$

$$x = \pm\frac{1}{2}.$$

① $1 = 8\lambda x$

$$\lambda = \frac{1}{8x} = \pm\frac{1}{4}$$

① $\Rightarrow$ $1 = 8x$

$$x = \frac{1}{8}$$

③ $4x^2 + y^2 = 1$

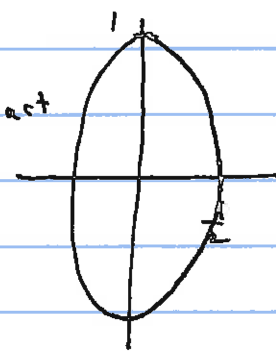
$$\frac{4}{64} + y^2 = 1$$

$$y^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

$$y = \pm\sqrt{15/16} = \pm\frac{\sqrt{15}}{4}$$

	$f = x + y^2$ (continuous)	
$(\frac{1}{2}, 0)$	$\frac{1}{2}$	
$(-\frac{1}{2}, 0)$	$-\frac{1}{2}$	min
$(\frac{1}{8}, \frac{\sqrt{15}}{4})$	$\frac{1}{8} + \frac{15}{16} = \frac{17}{16}$	max
$(\frac{1}{8}, \frac{-\sqrt{15}}{4})$	$\frac{1}{8} + \frac{15}{16} = \frac{17}{16}$	max

$4x^2 + y^2 = 1$



Compact

$\Downarrow$

max
min
exist.

Max value is $\frac{17}{16}$, attained at $\left(\frac{1}{8}, \pm\frac{\sqrt{15}}{4}\right)$

min value is $-\frac{1}{2}$, " " $\left(-\frac{1}{2}, 0\right)$.

Ex 3 Find the closest pt of the plane
$x+3y-2z=7$.
to the origin.

(Method I Section 1.5)
Method 2 L Multipliers

max/min $x^2+y^2+z^2$ ($=\text{distance}^2$ from $\vec{0}$)
constraint $x+3y-2z=7$.

$$\nabla(x^2+y^2+z^2) = (2x, 2y, 2z)$$
$$\nabla(x+3y-2z) = (1, 3, -2)$$

$$(2x, 2y, 2z) = \lambda.(1, 3, -2)$$

① $2x = \lambda$
② $2y = 3\lambda$
③ $2z = -2\lambda$
④ $x+3y-2z=7$.

solve

① $\Longrightarrow$ $x=\frac{\lambda}{2}$
② $\Longrightarrow$ $y=\frac{3}{2}\lambda$
③ $\Longrightarrow$ $z=-\lambda$.

④ $\Longrightarrow$ $7 = x+3y-2z = \frac{\lambda}{2}+3\left(\frac{3}{2}\lambda\right)-2(-\lambda)$

$$7 = \frac{\lambda}{2}+\frac{9}{2}\lambda+2\lambda = \lambda(7)$$
$$\lambda=1.$$

$x=\frac{1}{2}$, $y=\frac{3}{2}$, $z=-1$.

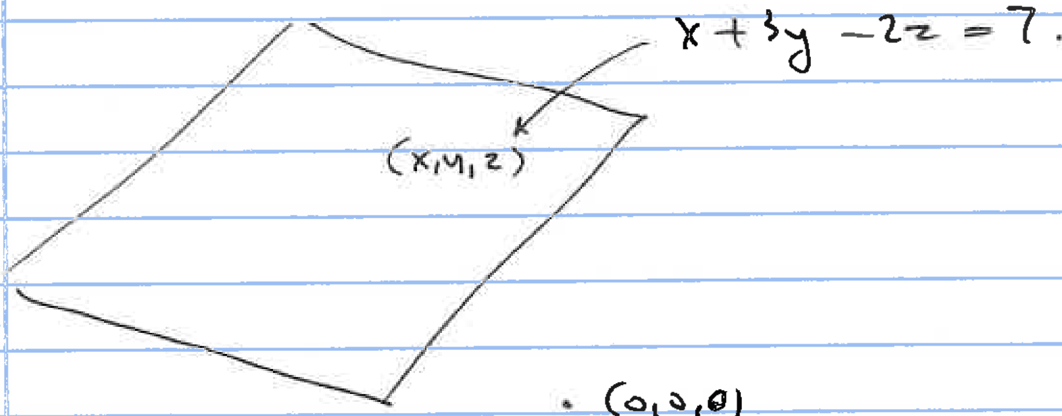
	$x^2+y^2+z^2$
$\left(\frac{1}{2}, \frac{3}{2}, -1\right)$	$\underbrace{\frac{1}{4}+\frac{9}{4}+1=\frac{14}{4}}_{\text{min}}$

· min
· no max.

dist $=\sqrt{\frac{14}{4}}=\frac{\sqrt{14}}{2}$.

Ex 3. Continue

Same problem without (LM 4.3.)
(As you would do it in 4.2)



$f = x^2+y^2+z^2 =$

$z = \dfrac{x+3y-7}{2}$

open

$h(x,y) = x^2+y^2+\left(\dfrac{x+3y-7}{2}\right)^2 \; : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\nabla h = \left(2x + 2\left(\frac{x+3y-7}{2}\right)^1 \cdot \frac{1}{2} , \; 2y + 2\cdot\left(\frac{x+3y-7}{2}\right)^1 \cdot \frac{3}{2} \right)$$

Ist D. Test $\nabla h = 0$

① $2x + \frac{1}{2}(x+3y-7) = 0$

② $2y + \frac{3}{2}(x+3y-7) = 0$

$$\begin{aligned} 5x + 3y &= 7 \\ 3x + 13y &= 21 \end{aligned} \xrightarrow{\text{soln}} \quad \begin{aligned} x &= \tfrac{1}{2} \\ y &= \tfrac{3}{2} \end{aligned}$$

① $-3(5x + 3y) = -21$

② $5(3x + 13y) = 105$

$$
\begin{aligned}
-15x - 9y &= -21 \\
15x + 65y &= 105 \\
\hline
56y &= 84
\end{aligned}
$$

$$y = \frac{84}{56} = \frac{3}{2}$$

$$x = \frac{1}{2}$$

$$z = -1.$$

$$H_h = \begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{13}{2} \end{bmatrix}$$

$$\Delta_1 = \frac{5}{2} > 0$$

$$\Delta_2 = \frac{65-9}{4} = 14 > 0$$

local min at $(x,y) = \left(\frac{1}{2}, \frac{3}{2}\right)$