March 30, 2017

4.2 B

Ex $f(x) = x^3 - 3x$ (From Calculus I)

- $f'(x) = 3x^2 - 3$

$f'(x) = 0 \iff 3(x^2 - 1) = 0$

$\iff x = \pm 1$ c.p.

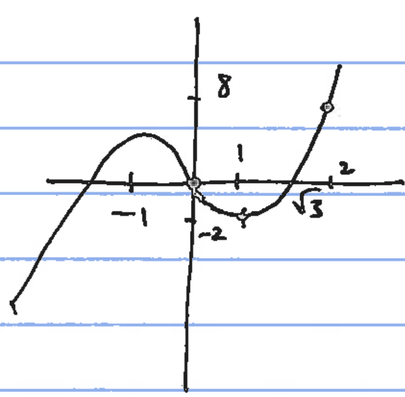
$f''(x) = 6x$

$f''(1) = 6$; 1 is local min

$f''(-1) = -6$, -1 is local max.

- Find max & min of $f = x^3 - 3x$ on $[0, 2]$.

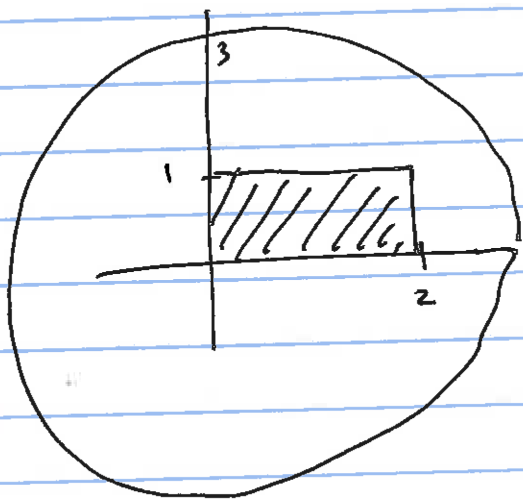
		$f(x) = x^3 - 3x$	
c.p.	1	$1 - 3 = -2$	smallest : min
not in $[0,2]$.	$\cancel{-1}$		
boundary	0	0	
boundary	2	$8 - 6 = 2$	largest : max



<u>Defn</u> A set $\mathbb{X}$ in $\mathbb{R}^n$ is called bounded if there is a sufficiently large Radius R s.t.

$$\mathbb{X} \subseteq \bar{B}_R(0) = \{ \vec{x} \mid \|\vec{x}\| \le R \}$$

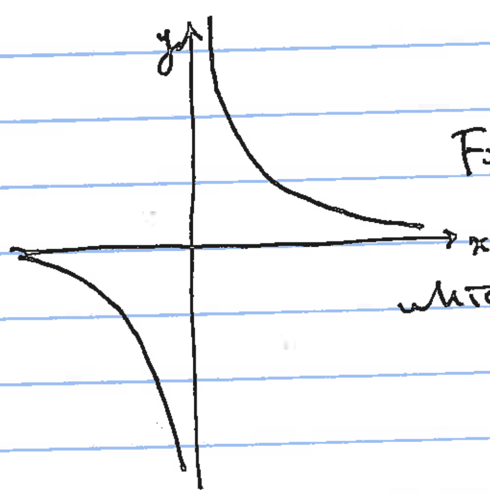
Ex (i) $\mathbb{X} = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 1\}$



$B_3(0)$ contains $\mathbb{X}$

$\mathbb{X}$ is bounded

(ii) $Y = \{(x,y) \mid xy = 1\}$



Y is not bounded since for no R large enough for which Y can be inside $B_R(0)$

Recall **Defn** A set $X \subseteq \mathbb{R}^n$ is called closed if X contains all of its boundary.

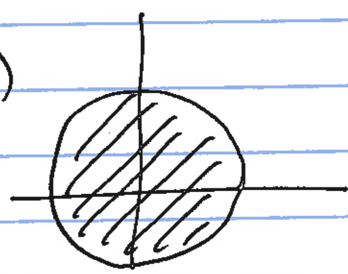
Defn A set $X \subseteq \mathbb{R}^n$ is called <u>compact</u> if
(1) X is bounded, and
(2) X is closed

Ex 1) $X = [a, b] \subseteq \mathbb{R}^1$ is compact, bounded, closed

2) $X = (a, b) \subseteq \mathbb{R}^1$ is not compact
bounded
not closed.

3) $X = \mathbb{R}$ closed.
not bounded
not compact

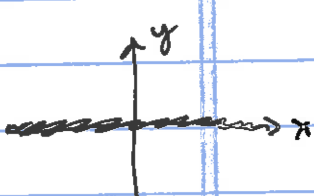
4)



$\{(x,y) \mid x^2+y^2 \leq 10\}$

closed, bounded, compact.

5) $\{(x,y) \mid x^2+y^2 < 10\}$
not closed
bounded
not compact



6) x-axis $\subseteq \mathbb{R}^2$; closed, unbounded, not compact

Thm: Extreme Value Theorem:

Let $f : \mathbb{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be s.t.

(i) f is continuous, and

(ii) $\mathbb{X}$ is compact. ($\underset{(a)}{\text{closed}}$ + $\underset{(b)}{\text{bounded}}$)

Then f must attain its maximum and minimum values on $\mathbb{X}$.

i.e. $\exists\ a_1, a_2 \in \mathbb{X}$,

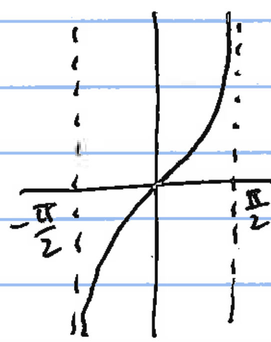
$$\forall x \in \mathbb{X} \qquad \underset{\text{min value}}{\underset{\parallel}{f(a_1)}} \leq f(x) \leq \underset{\text{max value}}{\underset{\parallel}{f(a_2)}}$$

Ex I

(i) ✓ • f cont

(iia) not $\mathbb{X}$ open / not closed

(iib) ✓ $\mathbb{X}$ bdd



$f = \tan x$

does not have max / min over $\underset{\mathbb{X}}{\underset{\parallel}{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}}$

II -

(i) ✓ • g cont

(iia) ✓ $\mathbb{X}$ closed

(iib) not $\mathbb{X}$ unbounded

$g(x) : \underset{\mathbb{X}}{\underset{\parallel}{\mathbb{R}}} \rightarrow \mathbb{R}$

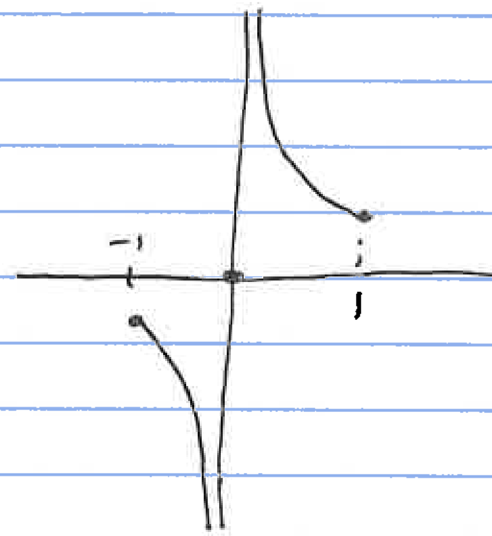
g has no max, no min

III

(i) no - h is not cont.

(iia) ✓ X closed $\Big\}$ compact

(iib) ✓ X bounded



$$h(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$h : [-1,1] \longrightarrow \mathbb{R}$.

no max / no min.

<u>Hence</u> there is no guarantee for max or min if any of the following fails.

(i) continuous

(iia) Closed domain

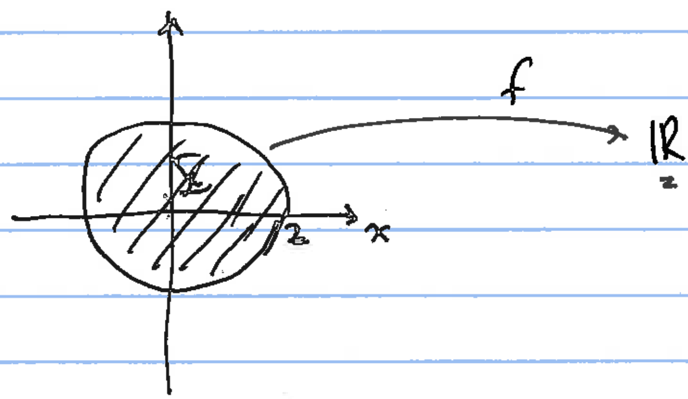
(iib) Bounded domain

Ex 1

$f(x,y) = (x-1)^2 + y^2$ (continuous)

Find max/min values of f over the region

$\underbrace{\underbrace{\mathbb{X} = \{(x,y) \mid x^2+y^2 \le 4\}}_{\text{closed + bounded}}}_{\text{compact domain}}$



Ext. V. Thm $\Rightarrow$ $\exists$ max/min.

plan:

- STEP 1 Find all interior c.p.
- STEP 2 Study f on the boundary.

STEP 1 $f = (x-1)^2 + y^2$

$\nabla f = (2(x-1), 2y)$

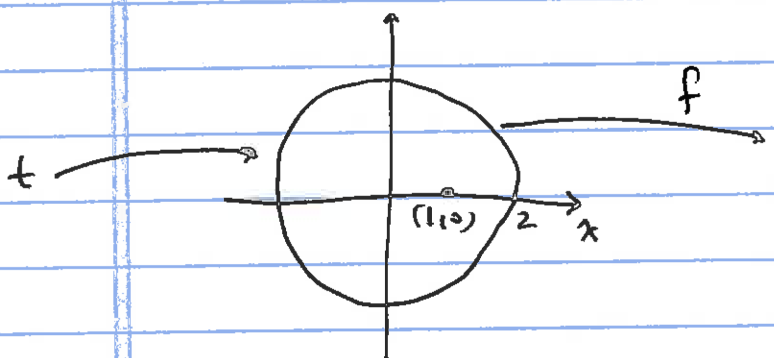
c.p. $\left.\begin{aligned} 2(x-1) &= 0 \\ 2y &= 0 \end{aligned}\right\}$ $(1,0)$

only interior critical pt.

__STEP 2__ Boundary $= \{(x,y) \mid x^2+y^2=4\}$

__A method:__ parametrize boundary

$$\gamma(t) = (\underbrace{2\cos t}_{x}, \underbrace{2\sin t}_{y}) \qquad 0 \le t \le 2\pi$$



$$g(t) = f(\underbrace{2\cos t}_{x}, \underbrace{2\sin t}_{y}) = \underbrace{(2\cos t - 1)^2}_{(x-1)^2} + \underbrace{(2\sin t)^2}_{y^2}$$

$$= 4\cos^2 t - 4\cos t + 1 + 4\sin^2 t$$

$$g(t) = \underbrace{5 - 4\cos t}_{\text{Find max/min of } g} \qquad 0 \le t \le 2\pi$$

$g'(t) = +4\sin t \qquad g'(t) = 0 \iff t = 0, \pi, 2\pi.$

	t	(x,y)	$f = (x-1)^2 + y^2$	
Boundary	$t=0$	$(2,0)$	1	
	$t=\pi$	$(-2,0)$	9	← max of f
	$t=2\pi$	$(2,0)$	1	
interior		$(1,0)$	0	← min of f

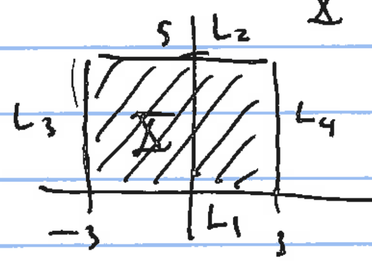
Exc #32 p276

$$f(x,y) = x^2 + xy + y^2 - 6y$$

Find max/min values of on $\underbrace{\left\{ \begin{array}{l} -3 \le x \le 3 \\ 0 \le y \le 5 \end{array} \right.}_{\mathbb{X}}$

- Continuity ✓
- $\mathbb{X}$ is closed
- $\mathbb{X}$ is bounded

$\left.\right\}$ compact



$\xRightarrow{\text{Ext V. Thm}}$ max/min exist on $\mathbb{X}$

STEP 1 Find all interior cp.

$$\nabla f = (2x + y,\; x + 2y - 6)$$

$$\nabla f = 0 \iff \left.\begin{array}{l} 2x + y = 0 \\ x + 2y = 6 \end{array}\right\} \qquad \begin{array}{l} x = -2 \\ y = 4 \end{array}$$

Only one interior cp. $(-2,4)$.

STEP 2 Boundary L_1 $(-3 \le x \le 3)$ & $y = 0$

L_2 $(-3 \le x \le 3)$ & $y = 5$

L_3 $x = -3$ & $(0 \le y \le 5)$

L_4 $x = 3$ & $(0 \le y \le 5)$

Study

$f = x^2 + xy + y^2 - 6y$ on each segment L_i:

L_1 $\left.\begin{array}{l} -3 \le x \le 3 \\ y = 0 \end{array}\right\}$

$f(x,0) = x^2$

$-3 \le x \le 3$

$\frac{d}{dx} f(x,0) = 2x = 0$

$x = 0$

$(0,0)$	
$(3,0)$	
$(-3,0)$	

L_2 $-3 \le x \le 3$

$y = 5$

$$f(x,5) = x^2 + 5x + 25 - 30$$
$$= x^2 + 5x - 5$$

$(-\frac{5}{2}, 5)$	
$(-3,5)$	
$(3,5)$	

$\frac{d}{dx} f(x,5) = 2x + 5 = 0$

$x = -\frac{5}{2}$

L_3 $0 \le y \le 5$

$x = -3$

$$f(-3,y) = 9 - 3y + y^2 - 6y$$
$$= y^2 - 9y + 9$$

$\frac{d}{dy} f(-3,y) = 2y - 9 = 0$

$y = \frac{9}{2}$

$(-3, \frac{9}{2})$	
$(-3, 0)$	
$(-3, 5)$	

L_4: $0 \le y \le 5$

$x = 3$

$f(3,y) = y^2 - 3y + 9$

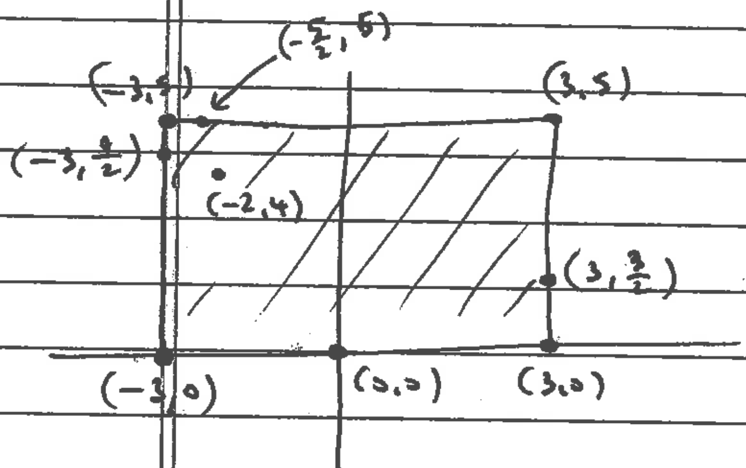
$\frac{d}{dy} f(3,y) = 2y - 3 = 0$ $\quad y = 3/2$

$(3, 3/2)$	
$(3, 0)$	
$(3, 5)$	

Combine all critical pts, boundary, corners

go Next page

	(x,y)	$f(x,y) = x^2 + xy + y^2 - 6y$	
✓	$(-2,4)$	-12	min value attained at $(-2,4)$
✓	$(0,0)$	0	
✓	$(-3,0)$	9	
✓	$(3,0)$	9	
✓	$(-\frac{5}{2},5)$	$-11\frac{1}{4}$	
✓	$(-3,5)$	-11	
corners Duplicated	$(3,5)$		
✓	$(-3,\frac{9}{2})$	$-11\frac{1}{4}$	
Duplicated	$(-3,0)$		
Duplicated	$(-3,5)$		
✓	$(3,3/2)$	$6\frac{3}{4}$	
Duplicated	$(3,0)$		
✓	$(3,5)$	19	max value attained at $(3,5)$



End of 4.2