March 10, 2017



4.2 Continue.

**Defn** Let $A$ be an $n \times n$ symmetric matrix in $\mathbb{R}$

$A$ is called positive definite if $\forall_{v \neq 0} v \in \mathbb{R}^n$ $v^T A v > 0$

$A$ is called negative definite if $\forall_{v \neq 0} v \in \mathbb{R}^n$ $v^T A v < 0$

$A$ is called indefinite if

- $\det A \neq 0$

and

- $\exists v, w \in \mathbb{R}^n$; $w^T A w < 0 < v^T A v$.

$A$ is degenerate if $\det A = 0$.

**Thm:** Let $f: \underset{\text{open}}{X} \subseteq \mathbb{R}^n \to \mathbb{R}^1$, $a \in X$
$f$ be twice continuously diff'ble on $X$
(at least around $a$)
Let $a$ be a critical pt; $\nabla f(a) = 0$.

Then:

If $H_f(a)$ is + definite then $f$ has a local min at $a$

If $H_f(a)$ is − definite " " " a local max at $a$

If $\left\{ \begin{array}{l} H_f(a) \text{ is indefinite} \\ (\operatorname{Det} H_f(a) \neq 0) \end{array} \right.$ " " " a saddle at $a$

If $\operatorname{Det} H_f(a) = 0$, then the test is inconclusive

Exc #12 p276

$$f(x,y) = e^{-x}(x^2+3y^2) : \mathbb{R}^2 \to \mathbb{R}^1$$

$0 \leq e^{-x}(x^2+3y^2)$ unbounded above

$f(0,y) = 3y^2 \nearrow \infty$ as $y \to \pm\infty$

Find all c.p. & classify.

$$\nabla f = \left(-e^{-x}(x^2+3y^2) + e^{-x} \cdot 2x \,,\; 6y e^{-x}\right)$$

① $+e^{x}(-x^2-3y^2+2x) = 0$
② $6y \cdot e^{-x} = 0.$
solve

② $\Longrightarrow$ $6y = 0$ since $e^{-x} > 0.$

$y = 0$

① $-x^2-3y^2+2x = 0$ since $e^{-x} > 0$

$-x^2+2x = 0$

$-x(x-2) = 0$

$x = 0$

or

$x = +2$

List c.p $\begin{cases}(0,0)\\(2,0)\end{cases}$

$$f_x = e^{-x}(-x^2 - 3y^2 + 2x)$$
$$f_y = 6ye^{-x}$$
$$f_{xx} = -e^{-x}(-x^2 - 3y^2 + 2x) + e^{-x}(-2x + 2)$$

$$H_f = \begin{bmatrix} e^{-x}(x^2 + 3y^2 - 2x - 2x + 2) & -e^{-x}6y \\ -e^{-x}6y & 6e^{-x} \end{bmatrix}$$

$$H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$ + definite $(0,0)$ is a local min

$$H_f(2,0) = \begin{bmatrix} e^{-2}(-2) & 0 \\ 0 & 6e^{-2} \end{bmatrix} = \underset{\substack{\vee \\ 0}}{e^{-2}} \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}$$

indefinite, Saddle at $(2,0)$

Ex

$$f(x,y,z) = 2x^2 + 3y^2 + 4z^2 + 2xy + 2xz - 4x - 2y - 2z.$$

Find all c.p. & classify

$$\left.\begin{aligned} f_x &= 4x + 2y + 2z - 4 \\ f_y &= 6y + 2x - 2 \\ f_z &= 8z + 2x - 2 \end{aligned}\right\} \underset{=}{\text{solve}} \Longrightarrow \begin{cases} x = 1 \\ y = 0 \\ z = 0 \end{cases}$$

Only One c.p.: $(1,0,0)$.

$$H_f = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 6 & 0 \\ 2 & 0 & 8 \end{bmatrix}$$ Is it +, −, indefinite?

$$\Delta_1 = 4 > 0$$

$$\Delta_2 = \begin{vmatrix} 4 & 2 \\ 2 & 6 \end{vmatrix} = 20 > 0$$

$$\Delta_3 = \begin{vmatrix} 4 & 2 & 2 \\ 2 & 6 & 0 \\ 2 & 0 & 8 \end{vmatrix} = (24\cdot 8 + 0 + 0) - (24 + 0 + 32)$$
$$= 192 - 56 = 136 > 0$$

$\Delta_1, \Delta_2, \Delta_3 > 0 \Longrightarrow$ + definite

$(1,0,0)$ local min.

PTO

Thm: Let $A$ be an $n \times n$ symmetric matrix (in $\mathbb{R}$)
$\det A \neq 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & & & a_{nn} \end{bmatrix}$$

$$\Delta_1 = a_{11}$$

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\vdots$

$1 \le k \le n$ $$\Delta_k = \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{k1} & \cdots & a_{kk} \end{vmatrix}$$

Then

$++++++$ - If all $\Delta_i > 0$, then $A$ is $+$ definite

$-+-+-+\cdots$ - If all $\Delta_{odd} < 0$, all $\Delta_{even} > 0$, then $A$ is $-$ definite

- Otherwise $A$ is indefinite

Ex $f = x^3 + xz^2 - 3x^2 + y^2 + 2z^2$

Find all c.p. & classify

$$\left.\begin{aligned} f_x &= 3x^2 + z^2 - 6x \\ f_y &= 2y \\ f_z &= 2xz + 4z \end{aligned}\right\} \quad \text{solve } \nabla f = 0$$

$$\left.\begin{aligned} ①\quad & 3x^2 + z^2 - 6x = 0 \\ ②\quad & 2y = 0 \\ ③\quad & 2xz + 4z = 0 \end{aligned}\right\}$$ solve. How? Not linear

② $\Rightarrow$ $y = 0$

③ $2xz + 4z = 0 = 2z(x+2)$

OR

$z = 0$

① $\Rightarrow$ $3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$

OR

$x = 2$

$x = -2$

① $\Rightarrow$ $12 + z^2 + 12 = 0$

$z^2 = -24$

no real solution.

List c.p. $(0,0,0)$
$(2,0,0)$

$$H_f = \begin{bmatrix} 6x-6 & 0 & 2z \\ 0 & 2 & 0 \\ 2z & 0 & 2x+4. \end{bmatrix}$$

$$H_f(0,0,0) = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\Delta_1 = -6$

$\Delta_2 = -12$ indefinite

$\Delta_3 = -48$

$(0,0,0)$ is a saddle

$$H_f(2,0,0) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$\Delta_1 = 6$

$\Delta_2 = 12$ + definite

$\Delta_3 = 96$ local min.