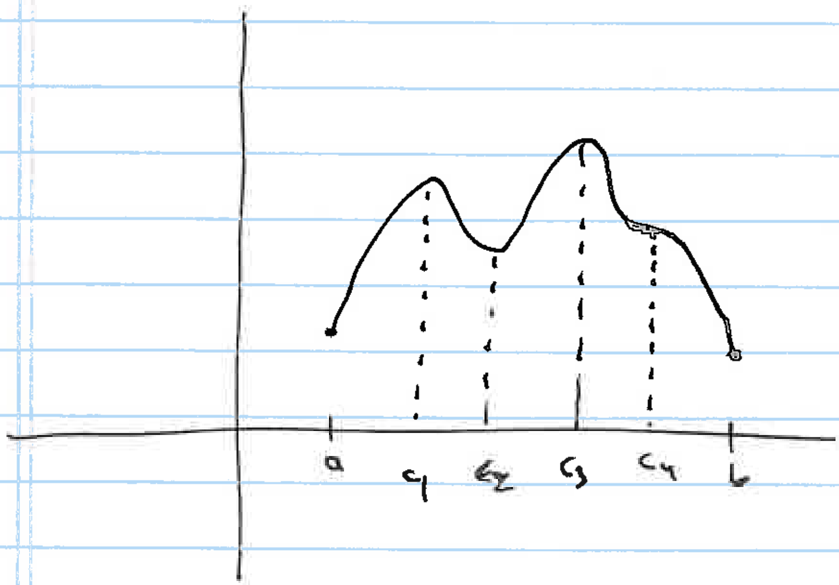
4.2

Calc I Review max/min

Recall $f:[a,b] \longrightarrow \mathbb{R}$



Solve $f'(x) = 0$ } $c_1, c_2, \dots$ critical pts.
when $f'(x)$ not defined }

Local Analysis: if f is twice diffble

$f'(c_i) = 0 \longrightarrow f''(c_i) > 0$ local min $\cup$

$\qquad\qquad\quad \searrow f''(c_i) < 0$ local max $\cap$

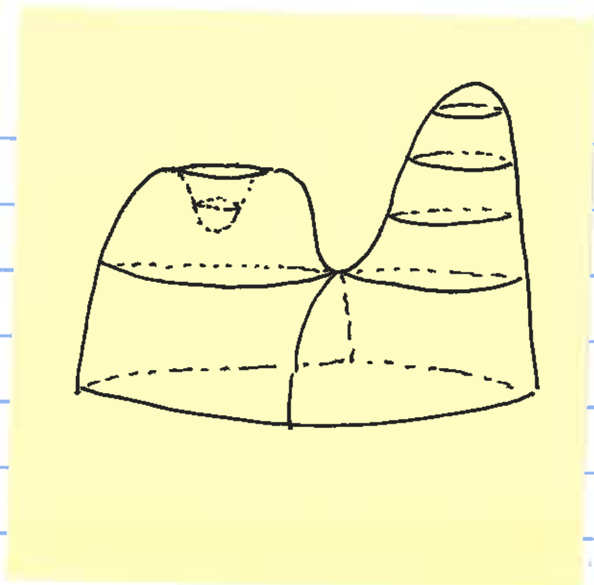
$\qquad\qquad\quad \searrow f''(c_i) = 0$ degenerate inconclusive

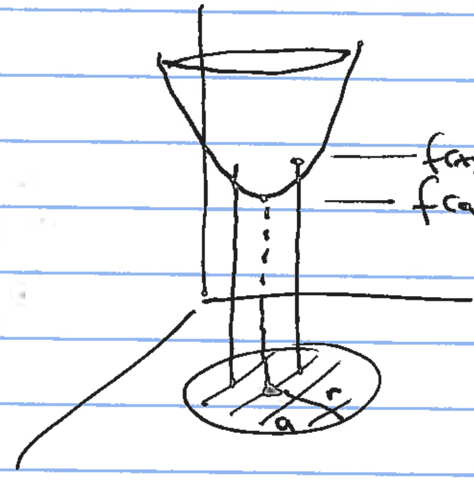
Global Analysis:

x	f
a	
b	
c_1	
c_2	
⋮	
⋮	

c.p. = $c_1, c_2, \dots$

To find max/min when Domain $= [a,b]$.





Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$,
Let $\vec{a} \in X$

- a is called a local minimum for f, if
  $\exists r > 0$, $f(\vec{x}) \geq f(\vec{a})$ for all $\vec{x} \in \{\vec{x} \mid \|\vec{x} - \vec{a}\| < r\}$
- a is called a local maximum for f, if
  $\exists r > 0$, $f(\vec{x}) \leq f(\vec{a})$ for all $\vec{x} \in \{\vec{x} \mid \|\vec{x} - \vec{a}\| < r\}$.
- a is called a global/absolute minimum if
  $f(\vec{x}) \geq f(\vec{a}) \quad \forall \vec{x} \in X = \text{domain}$
- a is called a global/absolute maximum if
  $f(\vec{x}) \leq f(\vec{a}) \quad \forall \vec{x} \in X$.

- $f(x) : X \to \mathbb{R}$ is called bounded (from) below,
  if $\exists M \quad f(x) \geq M \quad \forall x \in X$
- $f(x) : X \to \mathbb{R}$ is called bounded (from) above
  if $\exists N \quad f(x) \leq N \quad \forall x \in X$.
- f is called bounded, if bounded above & below.

Ex $f(x) = \frac{1}{x^2} : \mathbb{R} - \{0\}$. is bounded below

but f has no minimum.

THM : (First Derivative test)

Let $f : \underset{\text{open}}{\mathbb{X}} \subseteq \mathbb{R}^n \to \mathbb{R}^1$,

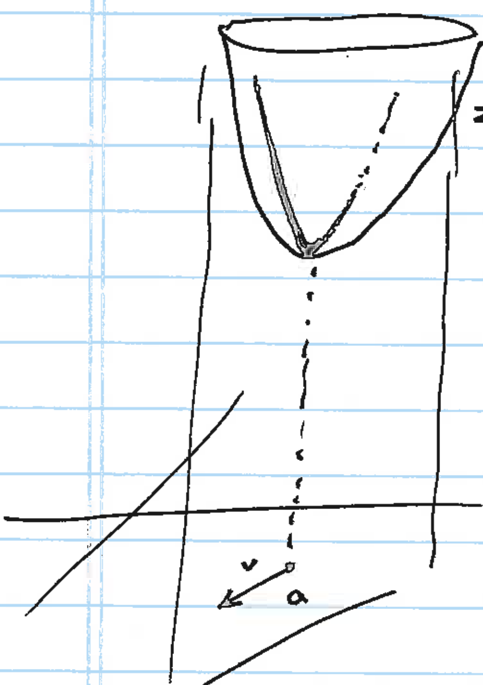
$a \in \mathbb{X}$, be a local extremum $\left(\begin{matrix}\text{local max} \\ \text{or} \\ \text{local min}\end{matrix}\right)$

Let f be diff'ble at a.

Then

$$\nabla f(a) = \vec{0} \quad \left(Df(a) = [0 \; 0 \, .. \, 0 \; 0] \right.$$

<u>Proof:</u> Case 1 a is a local minimum.



Let v be a direction in $\mathbb{R}^n$.

$$(D_v f)(a) = \lim_{h \to 0} \frac{f(a+hv) - f(a)}{h}$$

$$= \nabla f(a) \cdot v = 0, \text{ since}$$

$\left\{\begin{array}{l} a \text{ local min for } f \\ a \text{ is still local min for the slice obtained} \\ \text{by intersecting w/ plane thru } a, \parallel v \;\&\; z\text{-axis} \end{array}\right.$

$\forall v \in \mathbb{R}^n, \|v\| = 1 \quad D_v f(a) = \nabla f(a) \cdot v = 0$. So $\nabla f(a) = 0$ #

<u>Defn</u> Let $f: X \subseteq \mathbb{R}^n \to \mathbb{R}^1$.

$a \in X$ is called a critical point if

either • $\nabla f(a) = 0$

or • f is not diff'ble at a.

2nd Derivative test:

Let f be twice continuously diff'ble on X,

$f: X \to \mathbb{R}^1$

Let $a \in X$ be a critical pt $(\nabla f(a) = 0)$

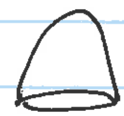
$$p_2 = f(a) + \underbrace{\cancel{\nabla f(a)}}_{0} \cdot (x-a) + \underbrace{\frac{1}{2}(x-a)^T H_f(a)(x-a)}_{\text{controlling local max / min / if } \det H_f(a) \neq 0}$$

Basic quadratic examples:

Examples 2×2

$$z = [x \;\; y] \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 3y^2 \geq 0$$

$$z = [x, y] \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -x^2 - 3y^2 \leq 0$$

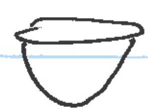


$$z = [x, y] \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 - y^2$$

$$z = [x\,y] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 - 2xy + 2y^2$$

$$= 2\left(x^2 - xy + \tfrac{1}{4}y^2 - \tfrac{1}{4}y^2 + y^2\right)$$

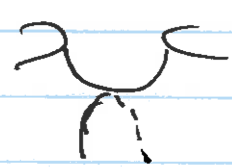
$$= 2\left(x - \tfrac{1}{2}y\right)^2 + \frac{3}{2}y^2 \geq 0$$



$$z = [x\,y] \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2 + 4xy$$

$$= x^2 + 4xy + 4y^2 - 3y^2$$

$$= (x + 2y)^2 - 3y^2$$



<u>Defn</u>

A symmetric 2×2 matrix $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$ is

called $+$ definite if $AC - B^2 > 0$, $A > 0$

called $-$ definite if $AC - B^2 > 0$, $A < 0$

called indefinite if $AC - B^2 < 0$.

called degenerate if $AC - B^2 = 0$.

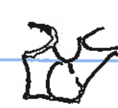
<u>Thm:</u> Let $f: X^{\text{open}} \subseteq \mathbb{R}^2 \to \mathbb{R}^1$

$a \in X$, f be twice continuously diff'ble around a, a be a critical pt: $\nabla f(a) = 0$

If $H_f(a)$ is $+$ definite, then f has a local min at a.

If $H_f(a)$ is $-$ definite then f has a local max at a.

If $H_f(a)$ is indefinite then f has a saddle at a



If $H_f(a)$ is degenerate, $\det H_f(a) = 0$; then this test is inconclusive.

Ex $f(x) = x^3 + 3x^2 + y^3 - 12y \; : \mathbb{R}^2 \to \mathbb{R}^1$

Find all critical pts and classify them.

$$\nabla f = (\overbrace{3x^2+6x}^{f_x}, \overbrace{3y^2-12}^{f_y}) = (0,0)$$

c.p. $\begin{cases} 3x^2+6x = 0 \quad \underline{\text{and}} \\ 3y^2 - 12 = 0 \end{cases}$

$3x^2+6x = 0$ $\qquad$ $3y^2-12=0$

$3x(x+2) = 0$ $\qquad$ $3(y^2-4)=0$

$\left(x = 0 \text{ OR } x = -2 \right)$ and $\left(y = 2 \text{ OR } y = -2 \right)$

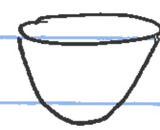
$\left.\begin{array}{l} (0,2) \\ (0,-2) \\ (-2,2) \\ (-2,-2) \end{array}\right\}$ all c.p.

H_f next page:

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x+6 & 0 \\ 0 & 6y \end{bmatrix}$$

$$H_f(0,2) = \begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix}$$

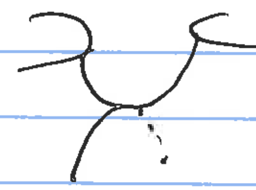
$\det = 72$
$A = 6 > 0$
$+$ definite



local min at $(0,2)$

$$H_f(0,-2) = \begin{bmatrix} 6 & 0 \\ 0 & -12 \end{bmatrix}$$

indefinite
$\det = -72$



saddle at $(0,-2)$

$$H_f(-2,2) = \begin{bmatrix} -6 & 0 \\ 0 & 12 \end{bmatrix}$$

indefinite
$\det = -72$

saddle at $(-2,2)$

$$H_f(-2,-2) = \begin{bmatrix} -6 & 0 \\ 0 & -12 \end{bmatrix}$$

$-$ definite
$\det = 72$
$-6 = A < 0$

local max. at $(-2,-2)$