March 6, 2017

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Chap IV

Review:

4.1 Taylor polynomials for 1 variable (Calc II)

$f: \underset{\overset{\psi}{a}}{\overset{\text{open}}{\text{Interval}}} \subseteq \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

$$p_1(x;a) = f(a) + f'(a)(x-a);$$

$$p_2(x;a) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2;$$

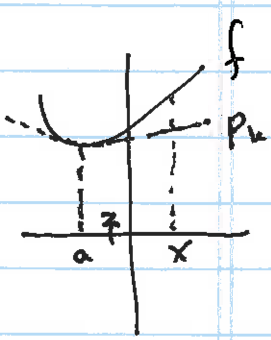
$$p_n(x;a) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k;$$

provided that f is n times diff'ble.

Error / Remainder term

$$R_k(x;a) = f(x) - p_k(x;a)$$

Taylor's Thm: ① $\lim_{x\to a} \frac{R_k(x;a)}{(x-a)^k} = 0$, provided that f is k-times diff'ble.



② If f is $k+1$ times diff'ble:

$$R_k(x;a) = \frac{f^{(k+1)}(z)}{(k+1)!}(x-a)^{k+1}$$ for some z between a & x.

Furthermore if $\left|f^{(k+1)}(x)\right| \le M$, then

$$\left|R_k(x;a)\right| \le \frac{M}{(k+1)!}(x-a)^{k+1} \quad (*)$$

Ex $\quad f(x) = e^x$

$f'(x) = e^x, \quad f^{(k)}(x) = e^x.$

$f^{(k)}(0) = 1.$

$$p_1(x;0) = 1 + x$$

$$p_2(x;0) = 1 + x + \frac{x^2}{2}$$

$$p_3(x;0) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$p_n(x;0) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Want to estimate $e^{0.04}$

If we use p_1: $e^{0.04} \approx 1.04$; what is an error estimate?

$(*)$ above $\Rightarrow$ $\left|R_1(x;0)\right| \le \frac{3}{2}(0.04)^2 = 0.0024.$

M can be taken as 3

$3 = M \ge \left| e^x \right|$ for $-1 < x < 1$

$x \approx 0$

p_1 estimates $e^{0.04}$ to be 1.04 ± 0.0024

If we use P_2

$$e^{0.04} \approx 1 + 0.04 + \frac{(0.04)^2}{2}$$

$$= 1.0408 \ (\pm \text{ error})$$

$$|R_2(x;0)| \le \frac{M}{3!}(x-a)^3$$

we can take $M=3$

$$(P_2 \text{ error}) \le \frac{3}{3!}(0.04)^3 = \frac{1}{2}(0.000064)$$

$$= 0.000032$$

$$e^{0.04} \approx \underbrace{1.0408}_{\text{approximation}} \pm \underbrace{0.000032}_{\text{error estimate}}$$

4.1 multivariable Taylor polynomials.

Recall 1^{st} degree Taylor polynomial is the tangent plane approximation

Defn. Let $f : X^{open} \subseteq \mathbb{R}^n \to \mathbb{R}^1$ be diffble at least twice

Gradient $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n} \right)$

First Deriv. $Df = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & & \frac{\partial f}{\partial x_n} \end{bmatrix}$. $1 \times n$

Total differential $df = \frac{\partial f}{\partial x_1} \cdot dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$.

Hessian

$$Hf = D(\nabla f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$n \times n$

observe that

$$\begin{cases} f: \mathbb{R}^n \to \mathbb{R}^1 \\ Df: \mathbb{R}^n \to \mathbb{R}^n \\ Hf: \mathbb{R}^n \to \mathbb{R}^{n^2} \end{cases}$$

<u>Ex 1</u> $f(x,y) = x^2 e^y + x - y$

$$Df = \begin{bmatrix} 2xe^y + 1 & x^2 e^y - 1 \end{bmatrix}$$

$$Hf = \begin{bmatrix} 2e^y & 2xe^y \\ 2xe^y & x^2 e^y \end{bmatrix}$$

$$df = (2xe^y + 1) \cdot dx + (x^2 e^y - 1) \cdot dy$$

We want p_1 and p_2 of f at $(x,y) = (3,0)$

$$f(3,0) = 12$$

$$Df(3,0) = \begin{bmatrix} 7 & 8 \end{bmatrix}$$

$$df(3,0) = 7dx + 8dy$$

$$Hf(3,0) = \begin{bmatrix} 2 & 6 \\ 6 & 9 \end{bmatrix}$$

$$P_1((x,y);(3,0)) = 12 + \begin{bmatrix} 7 & 8 \end{bmatrix} \begin{bmatrix} x-3 \\ y-0 \end{bmatrix}$$

Compare to Tangent plane approximation

$$= 12 + 7(x-3) + 8(y-0)$$

$$P_2((x,y);(3,0)) = \underbrace{12 + \begin{bmatrix} 7 & 8 \end{bmatrix} \begin{bmatrix} x-3 \\ y-0 \end{bmatrix}}_{P_1} + \frac{1}{2} \begin{bmatrix} x-3 & y-0 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} x-3 \\ y-0 \end{bmatrix}$$

$$= \underbrace{12 + 7(x-3) + 8(y-0)}_{P_1} + \frac{1}{2}\left(2(x-3)^2 + 12(x-3)(y-0) + 9(y-0)^2 \right)$$

$$P_2(x,y) = f(3,0) + \frac{\partial f}{\partial x}(3,0)(x-3) + \frac{\partial f}{\partial y}(3,0)(y-0) +$$

$$+ \frac{1}{2} \underbrace{\left[\frac{\partial^2 f}{\partial x^2}(3,0)(x-3)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(3,0)(x-3)(y-0) + \frac{\partial^2 f}{\partial y^2}(3,0)(y-0)^2 \right.}_{\text{in open form}}$$

<u>Defn</u> Let $f: \overset{open}{X} \subseteq \mathbb{R}^n \to \mathbb{R}^1$, $a \in X$
f be twice diffble.

First degree Taylor Polynomial

$$p_1(\underbrace{(x_1, .. x_n)}_{\vec{x}}; \underbrace{(a_1 .. a_n)}_{\vec{a}}) = f(\vec{a}) + \underset{1 \times n}{Df(\vec{a})} \cdot \underset{n \times 1}{(\vec{x} - \vec{a})^T}$$

2nd degree Taylor Polynomial.

$$p_2(\vec{x}, \vec{a}) = f(\vec{a}) + \underset{1 \times n}{Df(\vec{a})} \cdot \underset{n \times 1}{(\vec{x} - \vec{a})^T} + \left[\frac{1}{2} \underset{1 \times n}{(x-a)} \cdot \underset{n \times n}{Hf(a)} \cdot \underset{n \times 1}{(\vec{x} - \vec{a})^T}\right]$$

Ex $\quad f(x,y,z) = x^2 y + xyz + y$

p_1 & p_2 at $(x,y,z) = (1,2,3)$

$$f(1,2,3) = 2 + 6 + 2 = 10$$

$$\frac{\partial f}{\partial x} = 2xy + yz \qquad \frac{\partial f}{\partial x}(1,2,3) = 4 + 6 = 10$$

$$\frac{\partial f}{\partial y} = x^2 + xz + 1 \qquad \frac{\partial f}{\partial x}(1,2,3) = 1 + 3 + 1 = 5$$

$$\frac{\partial f}{\partial z} = xy \qquad \frac{\partial f}{\partial z}(1,2,3) = 2$$

$$p_1 = 10 + 10(x-1) + 5(y-2) + 2(z-3)$$

$$= 10 + \underbrace{\begin{bmatrix} 10 & 5 & 2 \end{bmatrix}}_{Df(1,2,3)} \cdot \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix}$$

Next Want p_2

$$Hf = \begin{bmatrix} 2y & 2x+z & y \\ 2x+z & 0 & x \\ y & x & 0 \end{bmatrix}$$

$$Hf(1,2,3) = \begin{bmatrix} 4 & 5 & 2 \\ 5 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$p_2 = 10 + \begin{bmatrix} 10 & 5 & 2 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \quad \end{bmatrix}$$

$$\begin{bmatrix} x-1 & y-2 & z-3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 2 \\ 5 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix}$$

$$p_2\big((x,y,z);(1,2,3)\big) = 10 + 10(x-1) + 5(y-2) + 2(z-3) + \cdots$$

$$+\frac{1}{2}\Big(4(x-1)^2 + 10(x-1)(y-2) + 4(x-1)(z-3) + 0(y-2)^2 + \cdots$$

$$\cdots + 2(y-2)(z-3) + 0\cdot(z-3)^2\Big)$$