Feb 27, 2017

2.6 Implicit Diff^n & Implicit Function Thm

Ex 1)

$$2x^2 + yz + z^3 = 4$$
$$xyz + z^2 - x^3y^2 = 1.$$

- Can you solve x & y in terms of z?
  Probably not.
- (a) Does there exist a diff'ble solution $(x, y) = g(z)$ locally near $(x,y,z) = (1,1,1)$?
  (b) If so, find $\frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}$ at $(1,1,1)$.

Sol^n

Let $F(x,y,z) = (2x^2 + yz + z^3,\ xyz + z^2 - x^3y^2)$

$F : \mathbb{R}^3 \to \mathbb{R}^2$

$$DF = \begin{bmatrix} 4x & z & y+3z^2 \\ yz - 3x^2y^2 & xz - 2yx^3 & xy + 2z \end{bmatrix}$$

$$DF(1,1,1) = \begin{array}{c} \begin{bmatrix} 4 & 1 & 4 \\ -2 & -1 & 3 \end{bmatrix} \\ \underbrace{x \quad\quad y}_{\text{want to solve}} \quad \underbrace{z}_{\text{in terms of } z.} \end{array}$$

$$Dg(1,1,1) = \begin{bmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{bmatrix}(1,1,1) = -\begin{bmatrix} 4 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

IMP F. Thm tells us that g exists locally

& $$Dg(1,1,1) = -\begin{bmatrix} 4 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{-2}\begin{bmatrix} -1 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -7 \\ 20 \end{bmatrix} = \begin{bmatrix} -7/2 \\ 10 \end{bmatrix}.$$

$$\frac{\partial x}{\partial z}(1,1,1) = -\frac{7}{2}$$

$$\frac{\partial \cancel{x}y}{\partial z}(1,1,1) = 10.$$

IMPLICIT FUNCTION THM

Given m equations in $n+m$ variables:

$$\left.\begin{aligned} F_1(x_1, \dots x_n, y_1 \dots y_m) &= c_1 \\ F_2(x_1 \dots x_n, y_1 \dots y_m) &= c_2 \\ &\vdots \\ F_m(x_1, \dots x_n, y_1 \dots y_m) &= c_m \end{aligned}\right\}$$

Want to solve $(y_1 \dots y_m) = y$ in terms of $(x_1 \dots x_n) = x$ localy

$$F(x_1, x_2 \dots x_n, y_1 \dots y_m) = (F_1, \dots F_m)$$

$$F : \overset{\text{open}}{A} \subseteq \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$$

Let F be continuously diffble on A.

Let $(\vec{a}, \vec{b}) = (a_1, a_2 \dots a_n; b_1 \dots b_m) = (c_1, c_2, \dots, c_m) = \vec{C}$.

$$\text{Let } DF = \left[\begin{array}{ccc|ccc} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} & \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & & & & \\ \frac{\partial F_m}{\partial x_1} & & \frac{\partial F_m}{\partial x_n} & \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{array}\right]$$

$$\underbrace{\qquad\qquad}_{\substack{F_x \\ m \times n}} \qquad \underbrace{\qquad\qquad}_{\substack{F_y \\ m \times m}}$$

If $(\det F_y)(\vec{a}, \vec{b}) \neq 0$, then there exists a local diffble solution $\vec{y} = g(\vec{x})$ of the equation $F(x,y) = c$, that is,

$$F(x, g(x)) = c \quad \text{is satisfied}$$

<u>and</u>

$$\underbrace{(Dg)(a,b)}_{m \times n} = - \underbrace{F_y^{-1}(a,b)}_{m \times m} \cdot \underbrace{F_x(a,b)}_{m \times n}.$$

Ex 2

given
$$\begin{cases} x_1^3 + x_2 y_1 + x_3 y_1^3 + y_1 y_2^2 = 10 \\ x_1^4 + x_3 y_2 + x_2 y_1^4 + y_1^2 y_2 = 16 \end{cases}$$

$$(x_1, x_2, x_3, y_1, y_2) = (2, 1, 0, 1, -1) = (a, b)$$

Does there exist $(y_1, y_2) = g(x_1, x_2, x_3)$ near $(2, 1, 0, 1, -1)$.

$$F(x_1, x_2, x_3, y_1, y_2) = \left(x_1^3 + x_2 y_1 + x_3 y_1^3 + y_1 y_2^2,\ x_1^4 + x_3 y_2 + x_2 y_1^4 + y_1^2 y_2\right)$$

$$DF = \begin{bmatrix} 3x_1^2 & y_1 & y_1^3 & x_2 + 3x_3 y_1^2 + y_2^2 & 2y_1 y_2 \\ 4x_1^3 & y_1^4 & y_2 & 4y_1^3 x_2 + 2y_1 y_2 & x_3 + y_1^2 \end{bmatrix}$$

2×5

$$DF(\underset{x_1}{2}, \underset{x_2}{1}, \underset{x_3}{0}, \underset{y_1}{1}, \underset{y_2}{-1}) = \begin{bmatrix} 12 & 1 & 1 & \overset{1+1}{2} & -2 \\ 32 & 1 & -1 & \underbrace{\underset{4-2}{2} \quad 1}_{\text{want to solve } y_1, y_2} \end{bmatrix}$$

$$\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = 6 \neq 0$$

Imp F Thm $\Longrightarrow$ Yes $\exists g$ s.t.

$(y_1, y_2) = g(x_1, x_2, x_3)$ locally,
and g is diff'ble.

You do not know what g is!

$$(Dg)(2,1,0,1,-1) = -\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 12 & 1 & 1 \\ 32 & 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} 1 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 12 & 1 & 1 \\ 32 & 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} 76 & 3 & -1 \\ 40 & 0 & -4 \end{bmatrix} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{bmatrix}$$

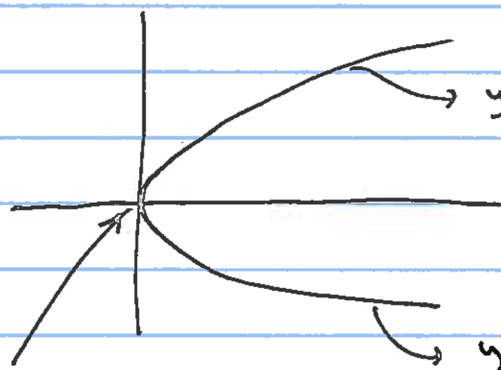
at $(2,1,0,1,-1)$.

$$\frac{\partial y_2}{\partial x_3}(2,1,0,1,-1) = \frac{-4}{-6} = \frac{2}{3}$$

Remark:

1. The solution may be local.
2. If Imp. F. Th. doesnot apply, there can be different reasons, why this method does not work.

Ex 1 $\quad x = y^2$



$y = \sqrt{x}$ for $y > 0$

$y = -\sqrt{x}$ for $y < 0$

about $(0,0)$, There is no solution of $y = g(x)$ in terms of x.

Compare: Ex 1, Ex 2

In 1, there is no solution about $(0,0)$.

In 2, there is a solution, but not diffble at $(0,0)$.

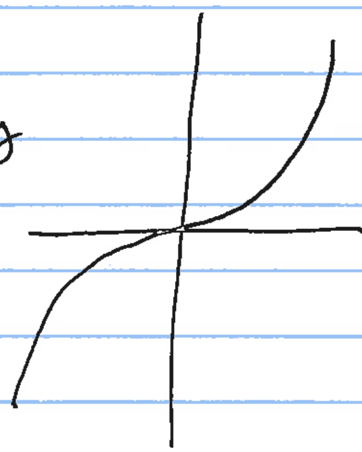
$$F(x,y) = x - y^2$$

$$DF = \begin{bmatrix} 1 & -2y \end{bmatrix}$$

$$DF(0,0) = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Imp F. Thm Doesnot apply.

Ex 2 $\quad x^3 = y$



$x = \sqrt[3]{y} = g(y)$

solution exists but not diffble at $(0,0)$.

Imp. F. Thm. doesnot apply.

$$G(x,y) = x^3 - y$$

$$DG = \begin{bmatrix} 3x^2 & -1 \end{bmatrix}$$

$$DG(0,0) = [0, -1]$$

Additional Example for finding Dg in variable form.

Ex 4

$$y^2 + u^4 + \sin u = 4$$
$$-y^3 + uy + e^{v} = -7.$$

Is (u,v) solvable near $(2,0,0)$ in terms of y?

$\exists ? g \quad (u,v) = g(y)$

If so what is $g'(y)$ near $(2,0,0)$?

$$DF = \begin{bmatrix} 2y & 4u^3 + \cos u & 0 \\ -3y^2 + u & y & e^{v} \end{bmatrix}$$

$$DF(2,0,0) = \begin{bmatrix} 4 & 1 & 0 \\ -12 & \underbrace{2 \quad\quad 1}_{\det \neq 0} \end{bmatrix}$$

$$\Rightarrow \exists g \qquad dg(2,0,0) = -\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -12 \end{bmatrix}$$

$$Dg = -\begin{bmatrix} 4u^3 + \cos u & 0 \\ y & e^{v} \end{bmatrix}^{-1} \begin{bmatrix} 2y \\ -3y^2 + u \end{bmatrix}$$

$$= -\frac{1}{e^{v}(4u^3 + \cos u)} \begin{bmatrix} e^{v} & 0 \\ -y & 4u^3 + \cos u \end{bmatrix} \begin{bmatrix} 2y \\ -3y^2 + u \end{bmatrix}$$

$$Dg(u,v,y) = -\frac{1}{e^{v}(4u^3 + \cos u)} \begin{bmatrix} e^{v} \cdot 2y \\ -2y^2 + (4y^3 + \cos u)(-3y^2 + u) \end{bmatrix}$$

Caution $(u,v) = g(y)$, but this formula doesn't give us Dg in terms of y only.