2.6 Continue

Thm: Let $f : \underline{X}^{\text{open}} \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^1$, $a \in \underline{X}$, $\nabla f(a) \neq \vec{0}$, f be diff'ble at a.

Then

(i) f increases fastest in the direction of $\nabla f(a)$, with a rate of $\|\nabla f(a)\|$;

(ii) f decreases fastest in the direction of $-\nabla f(a)$ with a rate of $-\|\nabla f(a)\|$;

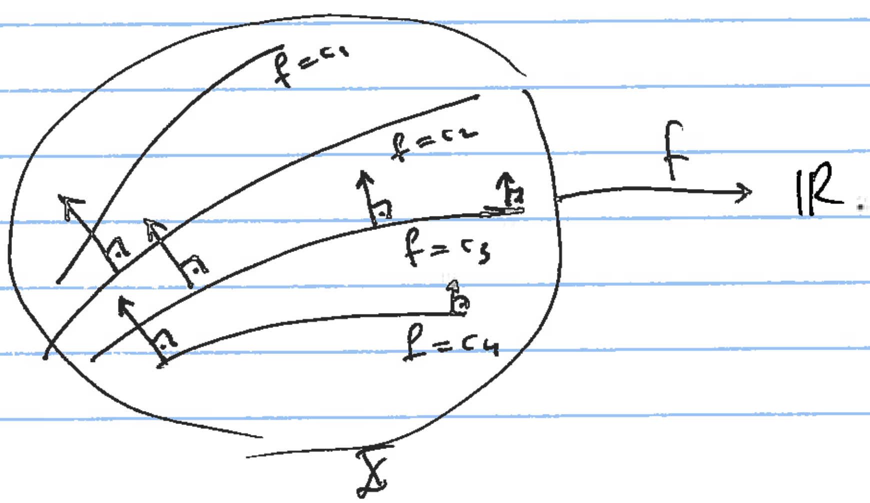
(iii) $\nabla f(a) \perp$ the level set of f thru a.

$$= \{x \in \underline{X} \mid f(x) = c = f(a)\}$$

(i), (ii) $(D_u f)(a) = \nabla f(a) \cdot u$ ← unit

$$= \|\nabla f(a)\| \cdot \underbrace{\|u\|}_{1} \cdot \cos\theta$$

(iii)



(iii)

$$0 = (D_u f)(a) = \nabla f(a) \cdot u = \underbrace{\|\nabla f(a)\|}_{\neq 0} \cdot \underbrace{\|u\|}_{\neq 0} \cos\theta$$

If one moves along a fixed level set, along which the function doesnot change. So $(D_u f)(a) = 0$, if u is tangential to the level set.

$\Rightarrow \quad \cos\theta = 0$

$\theta = \pm \frac{\pi}{2}$ (caution: $\pm$ only in $n=2$) $\theta \geq 0$ for $n \geq 3$.

Ex $\quad f(x,y,z) = x^2 y + yz + xz$

(a) Find the directions of steepest ascend / steepest descend at $(x,y,z) = (1,2,3)$

$$\nabla f = (2xy + z,\ x^2 + z,\ y + x)$$

$$\nabla f(1,2,3) = (7, 4, 3)$$

		Rate of change
steepest ascend direction :	$\dfrac{7i + 4j + 3k}{\sqrt{49+16+9}}$	$\sqrt{74}$
steepest descend direction :	$-\dfrac{7i + 4j + 3k}{\sqrt{49+16+9}}$	$-\sqrt{74}$

(b) Find an equation for the tangent plane to the surface

$$S = \{(x,y,z) \mid x^2y + yz + xz = 11\}$$

at $(1,2,3)$

$\nabla f(a) \perp S$ at a.

$(7,4,3) \perp S$ at a $(7,4,3) \perp$ Tangent plane at $\vec{a}$.

$$(7,4,3) \cdot \left[(x,y,z) - (1,2,3)\right] = 0$$

$$7x + 4y + 3z = 7 + 8 + 9 = 24$$

RULE   Let $f : X^{\text{open}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be diff'ble at a, $a \in X$, $\nabla f(a) \neq 0$

Then an equation for the tangent plane to the level set $\{\vec{x} \mid f(\vec{x}) = f(\vec{a})\}$ at $\vec{a}$ is given by

$$\overrightarrow{\nabla f(a)} \cdot \left[\vec{x} - \vec{a}\right] = 0$$

Exc #20 p 174.

Tangent plane to: $x^2 - 2y^2 + 5xz = 7$ at $(-1, 0, -\frac{6}{5})$.

(a) Old method (Tangent plane to an explicit graph)

$$5xz = 7 - x^2 + 2y^2$$

$$z = \frac{7 - x^2 + 2y^2}{5x} = f(x,y); \quad f(-1,0) = -\frac{6}{5}$$

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

$$\frac{\partial f}{\partial x} = \frac{(-2x)(5x) - 5 \cdot (7 - x^2 + 2y^2)}{25x^2}$$

$$\frac{\partial f}{\partial x}(-1,0) = \frac{-10 - 5(7-1)}{25} = \frac{-40}{25} = -\frac{8}{5}$$

$$\frac{\partial f}{\partial y} = \frac{4y}{5x}$$

$$\frac{\partial f}{\partial y}(-1,0) = 0$$

$$z = -\frac{6}{5} + -\frac{8}{5}(x+1) + 0(y-0)$$

$$5z = -6 - 8x - 8$$

$$\boxed{8x + 5z = -14}$$

New method

(b) $\{(x,y,z) \mid x^2 - 2y^2 + 5xz = 7\}$

$$F(x,y,z) = x^2 - 2y^2 + 5xz$$

$$\nabla F = (2x + 5z, -4y, 5x)$$

$$\nabla F(-1, 0, -\tfrac{6}{5}) = (-2-6, 0, -5)$$

$$= (-8, 0, -5)$$

$$\nabla F(\vec{a}) \cdot (\vec{x} - \vec{a}) = 0$$

$$(-8, 0, -5) \cdot ((x,y,z) - (-1, 0, -\tfrac{6}{5})) = 0$$

$$-8x - 5z = 8 + 0 + 6$$

$$\boxed{8x + 5z = -14}$$

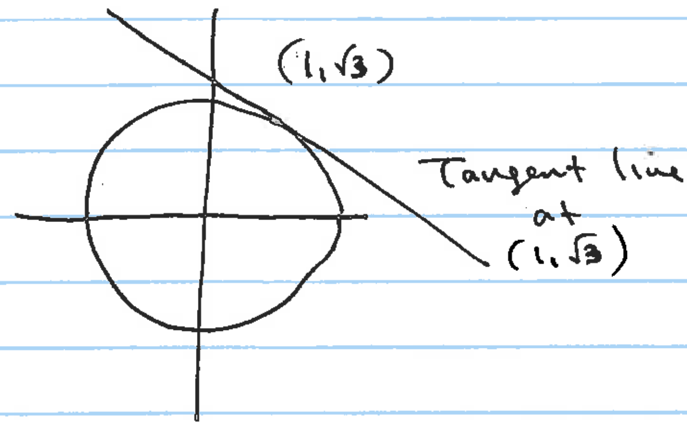
Compare (a) & (b).

2.6

C Implicit Diffⁿ.

Calculus I

$$x^2+y^2=4$$



$$\frac{d}{dx}\left(x^2+y^2\right)=2x+2y\cdot\frac{dy}{dx}=\frac{d}{dx}(4)=0$$

$$2x+2yy'=0$$

$$2yy'=-2x$$

$$y'=-\frac{2x}{2y}=-\frac{x}{y}$$

$y-b=m(x-a)$

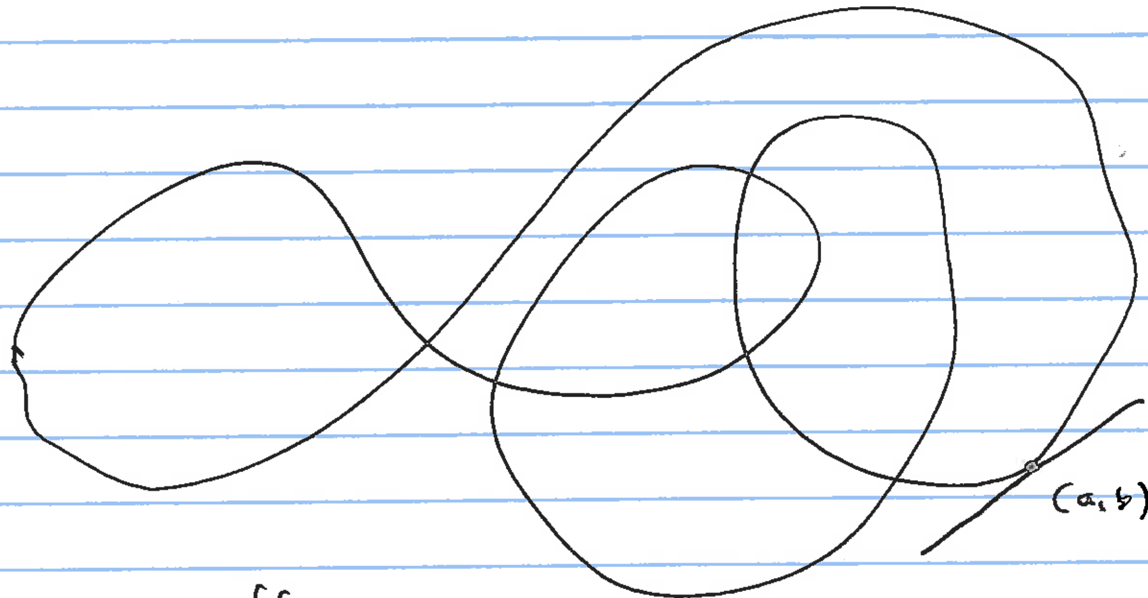
$$y'(1,\sqrt{3})=-\frac{1}{\sqrt{3}}$$

$$(y-\sqrt{3})=-\frac{1}{\sqrt{3}}(x-1)$$

$$y=-\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}+\sqrt{3}$$

General $n = 2$

$$\{(x,y) \mid F(x,y) = c\}$$



Implicit Diffn.

$$\frac{d}{dx} F(x,y) = \frac{d}{dx} F(x, y(x)) =$$

Assuming $y = y(x)$ in a diff'ble way

$$= F_x \cdot 1 + F_y \cdot \frac{dy}{dx} = \frac{d}{dx} c = 0$$

$$F_x + F_y y' = 0$$

$$y' = -\frac{F_x}{F_y}$$ valid if $F_y \neq 0$.

How do we do Implicit Differentiation for $n \geq 3$?

Ex 1 (A linear case)

$$F \quad \begin{cases} 2x + y + 3z = 1 \\ 5x + 3y - z = 2 \end{cases}$$

Solve for x & y in terms of z & calculate $\frac{\partial x}{\partial z}$, $\frac{\partial y}{\partial z}$

$$2x + y = 1 - 3z$$
$$5x + 3y = 2 + z.$$

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & +1 \\ 5 & 3 \end{bmatrix}^{-1} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1}}_{\text{inverse}} \underbrace{\begin{bmatrix} -3 \\ 1 \end{bmatrix}}_{(-)}$$

$$DF = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 3 & -1 \end{bmatrix}$$