Feb 23, 2017

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2.6

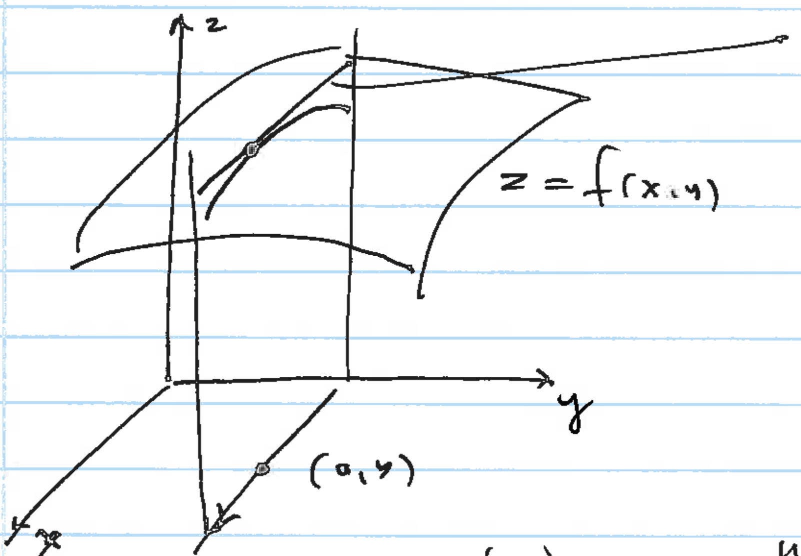
I Directional Derivatives

$$f : \underset{\text{open}}{X} \subseteq \mathbb{R}^n \to \mathbb{R}^1$$

$x_1, x_2 \ldots x_n$

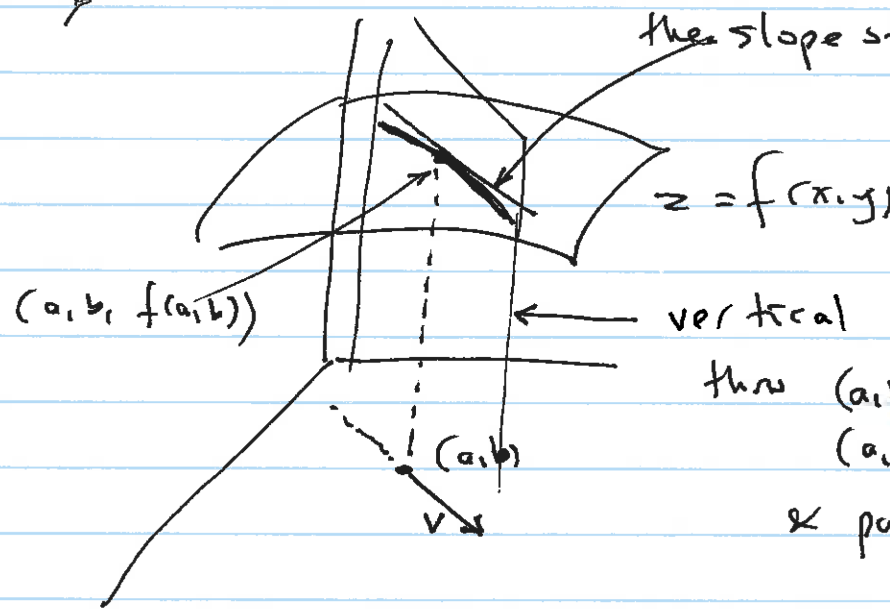
$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n} \right)$$

$$Df = \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \frac{\partial f}{\partial x_n} \right].$$



slope $= \frac{\partial f}{\partial x}(a,b)$

$\underbrace{\text{Directional derivative}}_{}$ in direction v, at (a,b) is the slope of the tangent line



vertical plane thru $(a,b,0)$, $(a,b,f(a,b))$ & parallel to v.

Defn $f: \overline{X}^{\text{open}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$

$\vec{a} \in \overline{X}$, v is a unit direction vector

$$(D_v f)(\vec{a}) = \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

if the limit exists.

Theorem: Let $f: \overline{X}^{\text{open}} \subseteq \mathbb{R}^n \to \mathbb{R}^1$

$\vec{a} \in \overline{X}$, $\|v\| = 1$, $v \in \mathbb{R}^n$

If f is diff'ble at $\vec{a}$, then

$$(D_v f)(\vec{a}) = \overrightarrow{\nabla f(a)} \cdot \vec{v}$$

Exc p173 #2

$$f(x,y) = e^y \sin x$$

$a = \left(\frac{\pi}{3}, 0\right)$ $\qquad u = \dfrac{3\hat{i} - \hat{j}}{\sqrt{10}}$ ← unit vector

Want: $D_u f(a)$

$$\nabla f = (e^y \cos x,\ e^y \sin x)$$

$$\nabla f\left(\frac{\pi}{3}, 0\right) = \left(e^0 \cdot \cos\frac{\pi}{3},\ e^0 \sin\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$(D_u f)(a) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \cdot (3, -1) \cdot \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}}\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)$$

Exx # 8 p173

$$f(x,y,z) = \frac{x e^y}{3z^2+1} \qquad \vec{a} = (2,-1,0)$$

$$u = i - 2j + 3k.$$

Want Directional derivative of f in the direction parallel to u, at a.

at $(2,-1,0)$

$$\frac{\partial f}{\partial x} = \frac{e^y}{3z^2+1} \qquad \frac{\partial f}{\partial x}(2,-1,0) = e^{-1} = \frac{1}{e}$$

$$\frac{\partial f}{\partial y} = \frac{x e^y}{3z^2+1} \qquad \frac{\partial f}{\partial y}(2,-1,0) = \frac{2}{e}$$

$$\frac{\partial f}{\partial z} = \frac{-6z x e^y}{(3z^2+1)^2} \qquad \frac{\partial f}{\partial z}(2,-1,0) = 0$$

$$\nabla f(2,-1,0) = \left(\frac{1}{e}, \frac{2}{e}, 0\right)$$

$$v = \frac{u}{\|u\|} = \frac{1}{\sqrt{14}}(1,-2,3)$$

$$(D_v f)(a) = \left(\frac{1}{e}, \frac{2}{e}, 0\right) \cdot (1,-2,3)\,\frac{1}{\sqrt{14}} = \frac{-3}{\sqrt{14}\,e}$$

Exc # 10 p173

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) $$\frac{\partial f}{\partial x}(0,0) = \frac{d}{dx} f(x,0)\Big|_{x=0} = \frac{d}{dx}\, 0 = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \frac{d}{dy} f(0,y)\Big|_{y=0} = \frac{d}{dy}\, 0 = 0.$$

$$\nabla f(0,0) = (0,0)$$

(b) $\vec{v} = a i + b j$ $a^2 + b^2 = 1$.   arbitrary unit vector in $\mathbb{R}^2$

$$(D_v f)(0,0) = \lim_{h \to 0} \frac{f((0,0) + h(a,b)) - f(0,0)}{h}.$$

$$= \lim_{h \to 0} \frac{f(ha, hb) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{ha \cdot hb}{\sqrt{(ha)^2 + (hb)^2}} - 0}{h} = \lim_{h \to 0} \frac{\dfrac{h^2 ab}{\sqrt{h^2}\sqrt{a^2+b^2}} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^2 ab}{|h|\, h} = \lim_{h \to 0} \frac{h}{|h|}\, ab =$$

$$\left\{\begin{array}{ll} \lim\limits_{h\to 0} \dfrac{h}{|h|}\, ab = 0 & \text{only when } ab = 0 \\[2ex] \lim\limits_{h\to 0} \dfrac{h}{|h|}\, ab \ \underline{\underline{\text{DNE}}} & \text{if } ab \neq 0 . \end{array}\right.$$

Briefly:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 \quad \text{since}$$

$$0 \leq \frac{|y|}{\sqrt{x^2+y^2}} \leq 1$$

$$0 \leq \left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq |x| \to 0$$

$$f(ah, bh) = \frac{ha\,hb}{\sqrt{(ha)^2+(hb)^2}} = \frac{h^2 ab}{\sqrt{h^2}\,\underbrace{\sqrt{a^2+b^2}}_{1}} = |h|ab .$$

f is continuous at $(0,0)$.

However, the sections of the graph through $(0,0)$ have $z = |h|\cdot c$ type corners about $(0,0)$ {when $ab \neq 0$}.

For Example: $a = b = 1/\sqrt{2}$: $\underbrace{f\left(\frac{1}{\sqrt{2}}h, \frac{1}{\sqrt{2}}h\right) = \frac{1}{2}|h|}_{\text{section of } f}$

we have different slopes.

$$\left\{\begin{array}{l} \lim\limits_{h\to 0^+} \dfrac{h}{|h|}\, ab = +1/2 \\[2ex] \lim\limits_{h\to 0^-} \dfrac{h}{|h|}\, ab = -1/2 . \end{array}\right.$$

$\leftarrow$ These are slopes towards the origin when $a = b = 1/\sqrt{2}$

In this last example

$(\nabla f)(0,0) = (0,0)$, but

$$\underbrace{(D_u f)(0,0)}_{\text{it doesn't exist when } ab \neq 0} \neq (\underbrace{\nabla f(0,0)}_{(0,0)}) \cdot (u) = 0$$

II If f is diff'ble at a:

$$(D_u f)(a) = \nabla f(a) \cdot u, \quad \text{(unit } u\text{)}$$

$$= |\nabla f(a)| \cdot \overbrace{|u|}^{1} \cdot \cos\theta$$

θ = angle between u & $\nabla f(a)$

$(D_u f)(a)$ is largest when $\theta = 0$, $\cos\theta = 1$.

$(D_u f)(a)$ is smallest when $\theta = \pi$, $\cos\theta = -1$.

steepest ascend [f increases fastest in the direction of $\nabla f(a)$ i.e. when $\theta = 0$.

steepest descend. [f decreases fastest in the direction of $-\nabla f(a)$ i.e. when $\theta = \pi$.