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1

2.5 Chain Rule

Ex 1 $g(t) = (t^2, t^3) = (x, y)$

$$z = f(x,y) = \cos x^2 y + \frac{x}{y}$$

$$\begin{tikzcd} & x \arrow[dr] & \\ t \arrow[ur] \arrow[dr] & & z \\ & y \arrow[ur] & \end{tikzcd}$$

$g \qquad f$

Book is in reverse order

$$\mathbb{R}^1 \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^1$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{dz}{dt} = \left((2xy)(-\sin x^2 y) + \frac{1}{y}\right)\cdot 2t + \left(\left((-x^2)\sin x^2 y\right) + \left(\frac{-x}{y^2}\right)\right)\cdot 3t^2$$

$$\left.\frac{dz}{dt}\right|_{t=2} = \left(64(-\sin 128) + \frac{1}{8}\right)\cdot 4 + \left(-16\sin 128 + \frac{-1}{16}\right)\cdot 12$$

$$t = 2 \implies x = t^2 = 4, \quad y = t^3 = 8$$

$$\frac{dz}{dt} = \left((2t^5)\cdot(-\sin t^7) + \frac{1}{t^3}\right)\cdot 2t + \left(-t^4 \sin t^7 - \frac{1}{t^4}\right)\cdot 3t^2$$

Ex 2

$$\overbrace{\begin{aligned} x &= u + 3v^2 \\ y &= u^2 - 4v \end{aligned}}^{F} \qquad \overbrace{\begin{aligned} s &= x^2 - 3xy \\ t &= x^3 + e^{y} \end{aligned}}^{G}$$

$$\begin{tikzcd} u \arrow[r] \arrow[dr] & x \arrow[r] \arrow[dr] & s \\ v \arrow[r] \arrow[ur] & y \arrow[r] \arrow[ur] & t \end{tikzcd}$$

$$\mathbb{R}^2 \xrightarrow[F]{} \mathbb{R}^2 \xrightarrow[G]{} \mathbb{R}^2$$

$$F(u,v) = (x,y) \qquad (s,t) = (G \circ F)(u,v)$$

$$G(x,y) = (s,t)$$

$$\frac{\partial s}{\partial u} = \frac{\partial s}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial s}{\partial v} = \frac{\partial s}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial t}{\partial u} = \frac{\partial t}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial t}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial t}{\partial v} = \frac{\partial t}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial t}{\partial y}\frac{\partial y}{\partial v}$$

$$\begin{bmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial \cancel{x}y}{\partial v} \end{bmatrix}$$

$$D(G \circ F) = DG \cdot DF$$

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial s}{\partial x} = 2x - 3y$$

$$\frac{\partial x}{\partial v} = 6v \qquad \frac{\partial s}{\partial y} = -3x$$

$$\frac{\partial y}{\partial u} = 2u \qquad \frac{\partial t}{\partial x} = 3x^2$$

$$\frac{\partial y}{\partial v} = -4 \qquad \frac{\partial t}{\partial y} = e^y$$

$$DF = \begin{bmatrix} 1 & 6v \\ 2u & -4 \end{bmatrix} \qquad DG = \begin{bmatrix} 2x-3y & -3x \\ 3x^2 & e^y \end{bmatrix}$$

$$(s,t) = (G \circ F)(u,v)$$

$$D(G \circ F) = DG \cdot DF$$

$$\begin{bmatrix} 2x-3y & -3x \\ 3x^2 & e^y \end{bmatrix} \begin{bmatrix} 1 & 6v \\ 2u & -4. \end{bmatrix}$$

$$= \begin{bmatrix} 2x - 3y - 6xu & (2x-3y)6v + 12x \\ 3x^2 + 2ue^y & 18x^2v - 4e^y \end{bmatrix}$$

$$\frac{\partial t}{\partial u}$$

Ex Ill-posed questions:

$$F(x,y,t) = x^2 t + x y t^2$$

$$x = t^2$$
$$y = t^3$$

$$\frac{d}{dt} F(x,y,t) = \,?$$

$$\begin{tikzcd} & x \arrow[dr] & \\ t \arrow[ur] \arrow[r] \arrow[dr] & y \arrow[r] & F \\ & \bar{t} \arrow[ur] & \end{tikzcd}$$

t has dual purpose

$$\frac{\partial x}{\partial t} = 2t \qquad \frac{\partial x}{\partial \bar{t}} = 0 \quad \left(x \,\&\, \bar{t} \text{ are independent} \right)$$

$$\frac{d}{dt} F(x,y,\bar{t}) = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial F}{\partial \bar{t}} \cdot \frac{\partial \bar{t}}{\partial t}$$

$$= (2x\bar{t} + y\bar{t}^2) \cdot 2t + (x\bar{t}^2) \cdot 3t^2 + (x^2 + 2xy\bar{t}) \cdot 1$$

$$F(x,y,\bar{t}) = x^2\bar{t} + xy\bar{t}^2$$

Then for the final answer take $t = \bar{t}$

$$\frac{d}{dt} F(x,y,t) = (2xt + yt^2) \cdot 2t + (xt^2) \cdot 3t^2 + (x^2 + 2xyt) \cdot 1$$