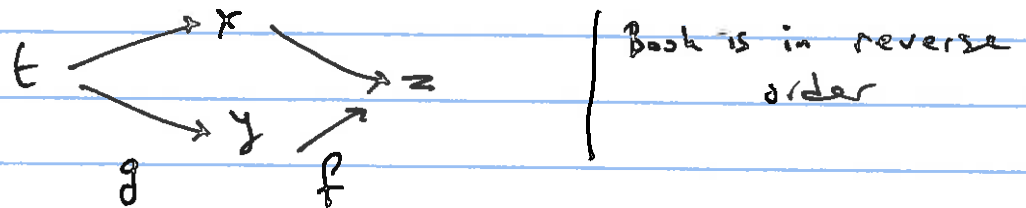


2.5 Chain Rule

Ex 1 $g(t) = (t^2, t^3) = (x, y)$
 $z = f(x, y) = \cos x^2 y + \frac{x}{y}$



$$\mathbb{R}^1 \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^1$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{dz}{dt} = \left((2xy)(-\sin x^2 y) + \frac{1}{y} \right) \cdot 2t + \left(\left((-x^2) \sin x^2 y \right) + \left(-\frac{x}{y^2} \right) \right) \cdot 3t^2$$

$$\frac{dz}{dt} \Big|_{t=2} = \left(64(-\sin 128) + \frac{1}{8} \right) \cdot 4 + \left(-16 \sin 128 + \frac{-1}{16} \right) \cdot 12$$

$$t=2 \implies x=t^2=4$$

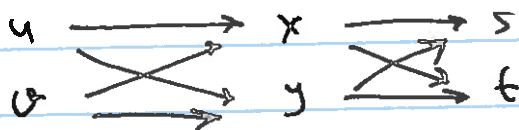
$$y=t^3=8$$

$$\frac{dz}{dt} = \left((2t^5) \cdot (-\sin t^7) + \frac{1}{t^3} \right) \cdot 2t + \left(-t^4 \sin t^7 - \frac{1}{t^4} \right) \cdot 3t^2$$

(2)

Ex 2

$$\begin{array}{l} \overbrace{x = u + 3v^2}^F \\ y = u^2 - 4v \end{array} \qquad \begin{array}{l} \overbrace{s = x^2 - 3xy}^G \\ t = x^3 + e^{xy} \end{array}$$



$$\mathbb{R}^2 \xrightarrow{F} \mathbb{R}^2 \xrightarrow{G} \mathbb{R}^2$$

$$F(u, v) = (x, y)$$

$$G(x, y) = (s, t)$$

$$(s, t) = (G \circ F)(u, v)$$

$$\frac{\partial s}{\partial u} = \frac{\partial s}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial s}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial s}{\partial v} = \frac{\partial s}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial s}{\partial y} \frac{\partial y}{\partial v}$$

$$\frac{\partial t}{\partial u} = \frac{\partial t}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial t}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial t}{\partial v} = \frac{\partial t}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial t}{\partial y} \frac{\partial y}{\partial v}$$

$$\begin{bmatrix} \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v} \\ \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$D(G \circ F) = DG \cdot DF$$

③

$$\frac{\partial x}{\partial u} = 1$$

$$\frac{\partial s}{\partial x} = 2x - 3y$$

$$\frac{\partial x}{\partial v} = 6v$$

$$\frac{\partial s}{\partial y} = -3x$$

$$\frac{\partial y}{\partial u} = 2u$$

$$\frac{\partial t}{\partial x} = 3x^2$$

$$\frac{\partial y}{\partial v} = -4$$

$$\frac{\partial t}{\partial y} = e^y$$

$$DF = \begin{bmatrix} 1 & 6v \\ 2u & -4 \end{bmatrix}$$

$$DG = \begin{bmatrix} 2x-3y & -3x \\ 3x^2 & e^y \end{bmatrix}$$

$$(s, t) = (G \circ F)(u, v)$$

$$D(G \circ F) = DG \cdot DF$$

$$\begin{bmatrix} 2x-3y & -3x \\ 3x^2 & e^y \end{bmatrix} \begin{bmatrix} 1 & 6v \\ 2u & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 2x-3y-6xu & (2x-3y)6v+12x \\ \underbrace{3x^2+2ue^y}_{\uparrow \frac{\partial t}{\partial u}} & 18x^2v-4e^y \end{bmatrix}$$

(4)

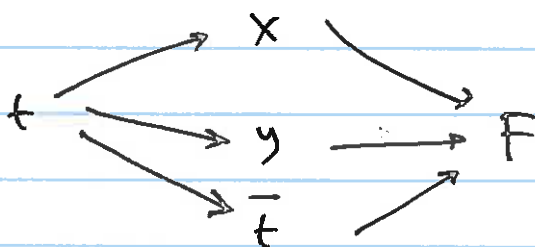
Ex Ill-posed questions:

$$F(x, y, t) = x^2 t + xy t^2$$

$$x = t^2$$

$$y = t^3$$

$$\frac{d}{dt} F(x, y, t) = ?$$



t has dual purpose

$$\frac{\partial x}{\partial t} = 2t$$

$$\frac{\partial x}{\partial \bar{t}} = 0 \quad (x \text{ \& } \bar{t} \text{ are independent})$$

$$\frac{d}{dt} F(x, y, \bar{t}) = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial F}{\partial \bar{t}} \cdot \frac{\partial \bar{t}}{\partial t}$$

$$= (2x\bar{t} + y\bar{t}^2) \cdot 2t + (x\bar{t}^2) \cdot 3t^2 + (x^2 + 2xy\bar{t}) \cdot 1$$

$$F(x, y, \bar{t}) = x^2 \bar{t} + xy \bar{t}^2$$

Then for the final answer ^{take} $t = \bar{t}$

$$\frac{d}{dt} F(x, y, t) = (2xt + yt^2) \cdot 2t + (xt^2) \cdot 3t^2 + (x^2 + 2xyt) \cdot 1$$