Feb 17, 2017

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2.3 Continue

Ex

$$L(x,y,z) = (2x-y+z,\ x+3z,\ y+3z)$$

$$\underset{3\times 3}{DL} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x-y+z \\ x+3z \\ y+3z \end{bmatrix}$$

For linear maps $F: \mathbb{R}^n \to \mathbb{R}^m$ DF is the same as the matrix of the linear transformation w.r.t. standard basis.

Ex $f: \mathbb{R}^3 \to \mathbb{R}^1$

x,y,z $\quad f = x^2y + e^x\cos z + y^3 z$

1x3 matrix $\quad Df = \begin{bmatrix} 2xy + e^x\cos z & x^2+3y^2z & -e^x\sin z + y^3 \end{bmatrix}$

vector $\quad \nabla f = (2xy + e^x\cos z,\ x^2+3y^2z,\ -e^x\sin z + y^3)$

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$$f(x,y,z) = (2x - y + 5z,\ x^2 + y,\ \ln yz)$$

$$Df = \begin{bmatrix} 2 & -1 & 5 \\ 2x & 1 & 0 \\ 0 & \frac{z}{yz} & \frac{y}{yz} \end{bmatrix}$$

$$Df(3,-1,-2) = \begin{bmatrix} 2 & -1 & 5 \\ 6 & 1 & 0 \\ 0 & -1 & -\frac{1}{2} \end{bmatrix}$$

Recall

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^1$$

x, y

Tangent plane at (a,b):

$$z = \underbrace{h(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\cdot(x-a) + \frac{\partial f}{\partial y}(a,b)\cdot(y-b)}_{\text{Tangent plane approx.}}$$

MORE GENERALLY:

$$F : \underset{\text{open}}{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$x_1, \ldots x_n$ $\qquad F = (F_1, F_2, \ldots F_m)$

$$\vec{a} \in X$$

$$\underbrace{\underbrace{H(\vec{x})}_{\substack{m\times 1 \\ \text{column} \\ \text{matrix}}} = \underbrace{F(a)}_{\substack{m\times 1 \\ \text{column} \\ \text{matrix}}} + \underbrace{\underbrace{DF(\vec{a})}_{(m\times n)}\cdot \underbrace{(\vec{x}-\vec{a})}_{(n\times 1)\text{ column}}}_{m\times 1 \text{ column matrix}}}_{\text{Tangent plane approximation to } F \text{ at } \vec{a}.}$$

Ex $F(x,y,z) = (x^2+y^2,\ xyz)$

Find the Tangent plane approximation of F at $(1,2,0)$.

$$F\left(\begin{bmatrix}1\\2\\0\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix}$$

$$\underset{2\times 3}{DF} = \begin{bmatrix} 2x & 2y & 0 \\ yz & xz & xy \end{bmatrix}$$

$$DF\left(\begin{bmatrix}1\\2\\0\end{bmatrix}\right) = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$H\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}5\\0\end{bmatrix} + \underset{2\times 3}{\begin{bmatrix} 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x-1 \\ y-2 \\ z-0 \end{bmatrix}}$$

$$= \begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix} 2(x-1) + 4(y-2) + 0(z-0) \\ 0(x-1) + 0(y-2) + 2(z-0) \end{bmatrix}$$

$$= \begin{bmatrix} 5 + 2x - 2 + 4y - 8 \\ 2z \end{bmatrix} = \begin{bmatrix} 2x + 4y - 5 \\ 2z \end{bmatrix}$$

#

Defn Let $F : \overline{X}^{\text{open}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$\vec{a} \in \overline{X}$.

F is called differentiable at a if

(i) All $\dfrac{\partial F_i}{\partial x_j}(\vec{a})$ exist, are finite

and

(ii) For $H(\vec{x}) = F(\vec{a}) + DF(\vec{a}) \cdot (\vec{x} - \vec{a})$,
a linear approximation of F, one has

$$\lim_{\vec{x} \to \vec{a}} \frac{\| F(\vec{x}) - H(\vec{x}) \|}{\| \vec{x} - \vec{a} \|} = 0.$$

2.4 Properties of derivatives

Let $f, g : X \subseteq \mathbb{R}^n \to \mathbb{R}^1$

$F, G : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$

- $\frac{\partial}{\partial x_i}(f+g) = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}$
- $\frac{\partial}{\partial x_i}(f \cdot g) = \frac{\partial f}{\partial x_i} \cdot g + f \cdot \frac{\partial g}{\partial x_i}$

$D(F+G) = DF + DG.$

Ex: $F(x,y) = (x^2+y,\ x-y,\ xy)$

$G(x,y) = (2x,\ y^2,\ x+y)$

$(F+G)(x,y) = (x^2+2x+y,\ y^2+x-y,\ x+y+xy)$

$$DF = \begin{bmatrix} 2x & 1 \\ 1 & -1 \\ y & x \end{bmatrix} \qquad DG = \begin{bmatrix} 2 & 0 \\ 0 & 2y \\ 1 & 1 \end{bmatrix}$$

$$D(F+G) = \begin{bmatrix} 2x+2 & 1 \\ 1 & 2y-1 \\ 1+y & 1+x \end{bmatrix}$$

See.

$D(F+G) = DF + DG$

Ex

$$F = (g, h)$$
$$f$$

} f, g, h are all real valued, all functions of x & y

$$f \cdot F = f(g, h) = (fg, fh)$$

$$D(f \cdot F) = D\big((fg, fh)\big)$$

$$= \begin{bmatrix} (fg)_x & (fg)_y \\ (fh)_x & (fh)_y \end{bmatrix}$$

$f_x = \frac{\partial f}{\partial x}$

$$= \begin{bmatrix} f_x \cdot g + f \cdot g_x & f_y \cdot g + f \cdot g_y \\ f_x \cdot h + f \cdot h_x & f_y \cdot h + f \cdot h_y \end{bmatrix}$$

$$= f \underbrace{\begin{bmatrix} g_x & g_y \\ h_x & h_y \end{bmatrix}}_{DF} + \underbrace{\begin{bmatrix} f_x \cdot g & f_y \cdot g \\ f_x \cdot h & f_y \cdot h \end{bmatrix}}_{2 \times 2}$$

$$= f \cdot DF + \underbrace{\begin{bmatrix} g \\ h \end{bmatrix}}_{2 \times 1} \underbrace{\begin{bmatrix} f_x & f_y \end{bmatrix}}_{1 \times 2}$$

$$= f \cdot DF + F \cdot Df$$

Formula in general Case

$$\left.\begin{array}{l} f: \mathbb{R}^n \to \mathbb{R}^1 \\ F: \mathbb{R}^n \to \mathbb{R}^m \end{array}\right\} \quad fF: \mathbb{R}^n \to \mathbb{R}^m$$

$$\underset{m \times n}{D(f \cdot F)} = \underbrace{\underset{\text{real.}}{f} \cdot \underset{m \times n}{DF}}_{m \times n} + \underbrace{\underset{m \times 1}{F} \cdot \underset{1 \times n}{Df}}_{m \times n.}$$

2.4 Second order Partials:

Ex $\quad f(x,y,z) = x^2y + y^3z$

$$\frac{\partial f}{\partial x} = 2xy$$

$$\frac{\partial f}{\partial y} = x^2 + 3y^2z$$

$$\frac{\partial f}{\partial z} = y^3$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6yz$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 3y^2$$

PTO

$$\frac{\partial^2 f}{\partial x\, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = 0$$

$$\frac{\partial^2 f}{\partial y\, \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = 3y^2$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = 0$$

Observe that for this example

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2x$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = 0$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = 3y^2$$