n = 1

$f : (a,b) \subseteq \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

$c \in (a,b)$

$f'(c)$ exists $\iff$ f is diff'ble at c

$f(x) = |x|$ $\not\Uparrow$ $\quad \Downarrow \quad \Downarrow$

f is continuous at c

n ≥ 2

$f : \overset{\text{open}}{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R} \qquad \vec{c} \in X$

$\frac{\partial f}{\partial x_i}(\vec{c})$ exists For all $i = 1, 2, \dots n$ $\Longleftarrow$ f is diff'ble at c

$\not\Longrightarrow$ see (ii)

see (i) $\not\Downarrow$ $\not\Uparrow$ see (ii) $\qquad \Downarrow$ $\not\Uparrow$ see (i)

f is continuous at c

Ex (ii) $f(x,y) = \begin{cases} xy/(x^2+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } x = y = 0 \end{cases}$

caution: order

(i)

$g(x,y) = \sqrt{x^2+y^2}$

Prop: Let $f : \underset{\text{open}}{X} \subseteq \mathbb{R}^n \to \mathbb{R}$, $c \in X$

f is diff'ble at $c \implies f$ is continuous at c.

Sketch (proof) in $\mathbb{R}^2$

$c = (a,b)$ f diff'ble at c: $\displaystyle \lim_{(x,y)\to(a,b)} \frac{\| f(x,y) - h(x,y) \|}{\| (x,y) - (a,b) \|} = 0$

where

$$h(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

$$\lim_{(x,y)\to(a,b)} \| f(x,y) - h(x,y) \| = \underbrace{\lim_{(x,y)\to(a,b)} \frac{\| f(x,y) - h(x,y) \|}{\| (x,y) - (a,b) \|}}_{\substack{0 \\ \text{diff'ble}}} \underbrace{\lim_{(x,y)\to(a,b)} \| (x,y) - (a,b) \|}_{0}$$

$$\lim_{(x,y)\to(a,b)} \left(f(x,y) - h(x,y) \right) = 0$$

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) =$$

$$= \lim_{(x,y)\to(a,b)} \left(\underset{\downarrow \, f(a,b)}{f(a,b)} + \underbrace{\frac{\partial f}{\partial x}(a,b)(x-a)}_{\substack{\downarrow \;\; x \to a \\ 0}} + \underbrace{\frac{\partial f}{\partial y}(a,b)(y-b)}_{\substack{\downarrow \;\; y \to b \\ 0}} \right)$$

So $\displaystyle \lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$; continuous at (a,b).

How do we see if a function is diff'ble?

Harder to use.

Thm: Let $f: \underline{X}^{open} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$.

If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are continuous on $\bar{X}$, then f is diff'ble on $\bar{X}$.

Easier to use

Thm: Let $f, g: \bar{X}^{open} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, and both be diff'ble at $(a,b) \in \bar{X}$.

Then (i) $f+g$ is diff'ble at (a,b)

(ii) $f \cdot g$ " " " "

(iii) $f-g$ " " " "

(iv) $c \cdot f$ " " " " $\forall c \in \mathbb{R}$

(v) f/g " " " ", if $g(a,b) \neq 0$ $(g(x,y) \neq 0$ near $(a,b))$

Thm: (informally)

Compositions of diff'ble functions are diff'ble.

(We'll see it in 2.5)

#36 $f(x,y) = \left(\frac{xy^2}{x^2+y^4}, \frac{x}{y} + \frac{y}{x} \right)$

D = Domain of $f = \{(x,y) \mid x \neq 0 \text{ and } y \neq 0\}$

$\left.\begin{array}{l} x \text{ is diff'ble on } D \\ y^2 \text{ is diff'ble on } D \end{array}\right\} \Rightarrow xy^2$ is diff'ble on D

x^2+y^4 " " $\Big\} \Rightarrow \frac{xy^2}{x^2+y^4}$ is diff'ble on D.

$\frac{x}{y}$ diff'ble on D since x & y are, $y \neq 0$

$\frac{y}{x}$ " " " y & x are, $x \neq 0$

$\frac{x}{y} + \frac{y}{x}$ " " " $\frac{x}{y}, \frac{y}{x}$ are on D,

$\left(\frac{xy^2}{x^2+y^4}, \frac{x}{y} + \frac{y}{x} \right)$ is diff'ble since each component is a diff'ble function of (x,y) on D.

Defn Let $f: X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1$

$x_1, x_2, \ldots x_n$

we define the gradient of f

$$\nabla f = \underbrace{\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots \frac{\partial f}{\partial x_n} \right)}_{}$$

n-components, $n = \#$ variables.

Defn Let $F(x_1, \ldots x_n) = (F_1, F_2, \ldots F_m)$

$F: X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

n variables ; m components

The first derivative matrix of f

$$DF = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} & & \frac{\partial F_2}{\partial x_n} \\ \vdots & & & & \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

Each row has the same component function

$m \times n$ matrix

Each column has the same differentiating variable

Differentiator : $\frac{\partial}{\partial x_i}$, (a differential operator)

Ex 1 (a)

$$F(\underbrace{x, y, z}_{\text{3 variables}}) = \underbrace{(x^2 + y^3 - 3z,\ 3xy^2)}_{\text{2 components}}$$

$$\underset{2\times 3}{DF} = \begin{bmatrix} 2x & 3y^2 & -3 \\ 3y^2 & 6xy & 0 \end{bmatrix}$$

(b)

$$G(\underbrace{x, y}_{\text{2 variables}}) = \underbrace{\left(e^x \cos y,\ \sin(y^2 x),\ e^{xy}\right)}_{\text{3 components}}$$

$$\underset{3\times 2}{DG} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ (\cos(y^2 x)) \cdot y^2 & 2xy\cos(y^2 x) \\ ye^{xy} & xe^{xy} \end{bmatrix}$$