Alternate Defn ② 2.3 Continue

Let $f: \underset{x_1,\dots x_n}{\mathbb{X}} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$; where $\mathbb{X}$ is an open subset of $\mathbb{R}^n$

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \dots x_n) =$$

$$= \lim_{h \to 0} \frac{f(x_1, x_2, \dots x_{i-1}, x_i + h, x_{i+1}, \dots x_n) - f(x_1, x_2, \dots, x_n)}{h}$$

when it exists.

($\Longleftrightarrow$ Definition we gave earlier for $D_{x_i} f = \frac{\partial f}{\partial x_i} = f_{x_i}$

"informally differentiating w.r.t. x_i while keeping other variables constant.")

Differentiability

Defn Let $f: \underbrace{\mathbb{X}}_{\text{open in } \mathbb{R}^2} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(a,b) \in \mathbb{X} = \text{domain}(f)$

f is called differentiable at (a,b) if

(1) $\frac{\partial f}{\partial x}(a,b)$, $\frac{\partial f}{\partial y}(a,b)$ both exist and are finite

and

(2) The linear function

$$h(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

is a "good" approximation of $f(x,y)$ ...

around (a,b), that is:

$$\lim_{(x,y)\to(a,b)} \frac{\|f(x,y)-h(x,y)\|}{\|(x,y)-(a,b)\|} = 0.$$ (This is what "good" is.)

Also: $z = h(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$

is called the tangent plane approximation of f at (a,b).

Also known as the first degree Taylor polynomial of f at (a,b).

Tangent plane to f at (a,b):

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

Ex1 $\quad f(x,y) = x^2 + y^2 \; : \; \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\vec{a} = (1,3) \qquad f(1,3) = 10$

$\dfrac{\partial f}{\partial x} = 2x \qquad \dfrac{\partial f}{\partial x}(1,3) = 2$

$\dfrac{\partial f}{\partial y} = 2y \qquad \dfrac{\partial f}{\partial y}(1,3) = 6.$

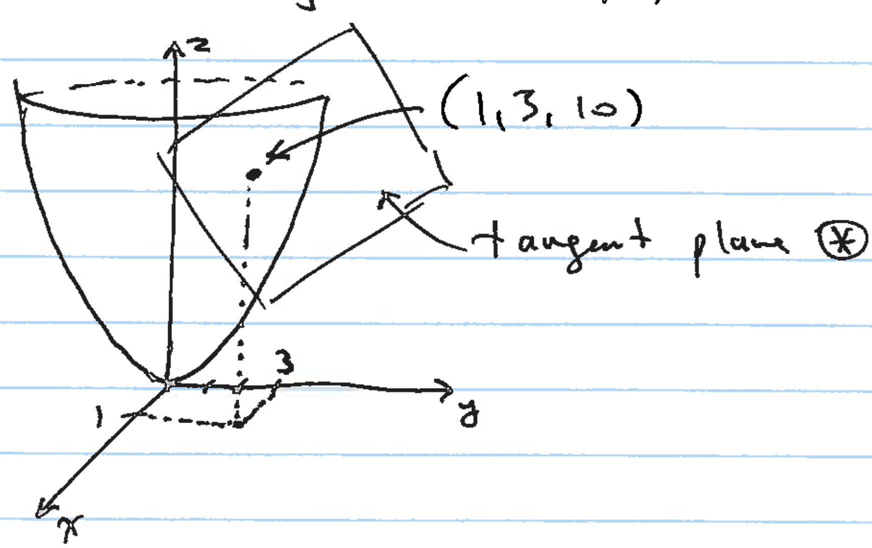
A

$$h(x,y) = f(1,3) + \frac{\partial f}{\partial x}(1,3)(x-1) + \frac{\partial f}{\partial y}(1,3)(y-3)$$

$$\underbrace{h(x,y) = 10 + 2(x-1) + 6(y-3)}_{\text{Tangent plane approximation of } f \text{ about } (1,3)}$$

⊛ $\quad z = 10 + 2(x-1) + 6(y-3)$

is the tangent plane to the graph of $f(x,y) = x^2 + y^2$ at $(1,3)$



B Why is this approximation "good"?

WTS "want to show"
$$\lim_{(x,y)\to(a,b)} \frac{\|f(x,y)-h(x,y)\|}{\|(x,y)-(a,b)\|} = 0$$

Sol^n

$f(x,y) = x^2 + y^2$

$h(x,y) = 10 + 2x - 2 + 6y - 18 = 2x + 6y - 10$

$$\lim_{(x,y)\to(1,3)} \frac{\|(x^2+y^2) - (2x+6y-10)\|}{\sqrt{(x-1)^2+(y-3)^2}}$$

$$= \lim_{(x,y)\to(1,3)} \frac{|x^2 - 2x + y^2 - 6y + 10|}{\sqrt{(x-1)^2+(y-3)^2}}$$

$$= \lim_{(x,y)\to(1,3)} \frac{(x - 2x + 1) + (y^2 - 6y + 9)}{\sqrt{(x-1)^2+(y-3)^2}}$$

$$= \lim_{(x,y)\to(1,3)} \frac{(x-1)^2 + (y-3)^2}{\sqrt{(x-1)^2+(y-3)^2}}$$

$$= \lim_{(x,y)\to(1,3)} \sqrt{(x-1)^2+(y-3)^2} = 0 \quad \checkmark$$

(c) Approximate $(1.003)^2 + (3.004)^2$
by using tangent plane approximation.

$$f = x^2 + y^2$$
$$h = 2x + 6y - 10 = 10 + 2(x-1) + 6(y-3)$$

$$(1.003)^2 + (3.004)^2 \approx 10 + 2(0.003) + 6(0.004)$$
$$= 10 + 0.006 + 0.024$$
$$= 10.030$$

$$0.00003 \geq |\text{error}| = |f(1.003, 3.004) - h(1.003, 3.004)|$$

Every approximation should be followed by an estimation of error.
(we need to learn that in 4.1)

Ex (2) $\sqrt{x^2+y^3}$

- Find the tangent plane at $(1,2)$
- Approximate $\sqrt{(1.04)^2+(1.999)^3}$ by the tangent plane approximation

$$\frac{\partial f}{\partial x} = \frac{2x}{2\sqrt{x^2+y^3}} \qquad \frac{\partial f}{\partial x}(1,2) = \frac{1}{\sqrt{1+8}} = \frac{1}{3}$$

$$\frac{\partial f}{\partial y} = \frac{3y^2}{2\sqrt{x^2+y^3}} \qquad \frac{\partial f}{\partial y}(1,2) = \frac{12}{2\cdot 3} = 2$$

$$f(1,2) = 3$$

$$\underbrace{h(x,y) = \overbrace{3 + \frac{1}{3}(x-1) + 2(y-2) = z}^{\text{Tangent plane}}}_{\text{T.P approx}}$$

$$\sqrt{(1.04)^2+(1.999)^3} \approx 3 + \frac{1}{3}(0.04) + 2(-0.001)$$

$$= 3 + 0.01\overline{3} - 0.002$$

$$= 3.011\overline{3}$$

Ex $f(x,y) = \begin{cases} \dfrac{x \cdot y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } x=y=0. \end{cases}$

Formally: $\dfrac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{\frac{0}{0+h^2} - 0}{h} = \lim_{h\to 0} \dfrac{0}{h} = 0.$

informally

$$\frac{\partial f}{\partial x}(0,0) = \frac{d}{dx} f(x,0)\Big|_{x=0} = \frac{d}{dx} 0 \Big|_{x=0} = 0$$

including when $x=0$

$$\frac{\partial f}{\partial y}(0,0) = \frac{d}{dy} f(0,y)\Big|_{y=0} = \frac{d}{dy} 0 \Big|_{y=0} = 0$$

including when $y=0$

if $(x,y) \neq (0,0)$: $\dfrac{\partial f}{\partial x} = \dfrac{y\cdot(x^2+y^2) - 2x \cdot xy}{(x^2+y^2)^2} = \dfrac{y^3 - x^2 y}{(x^2+y^2)^2}$

if $(x,y) \neq (0,0)$: $\dfrac{\partial f}{\partial y} = \dfrac{x^3 - y^2 x}{(x^2+y^2)^2}$

① $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE. ② f is not continuous at $(0,0)$

③ $\lim_{(x,y)\to(0,0)} \dfrac{\partial f}{\partial x}$ DNE; but $\dfrac{\partial f}{\partial x}(0,0) = 0$

$\lim_{(x,y)\to(0,0)} \dfrac{\partial f}{\partial y}$ DNE; but $\dfrac{\partial f}{\partial y}(0,0) = 0.$