Feb 10

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5.2 Continuity

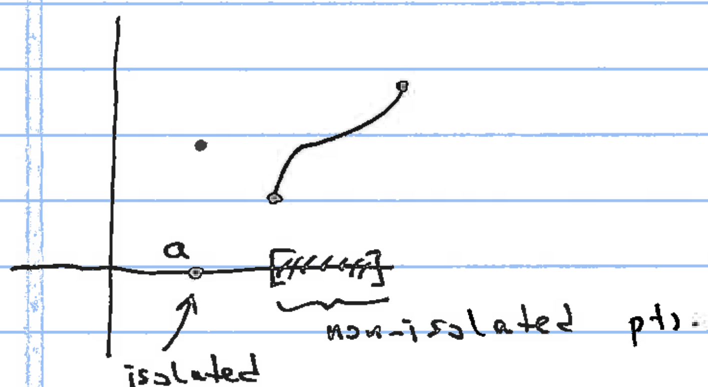
Defn Let $f : \underline{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $a \in \underline{X} =$ domain.

f is called continuous at a either

1) if a is an isolated pt of the domain

OR

2) $\lim\limits_{x \to a} f(x) = f(a)$, if a is not an isolated pt of the domain.



$\Updownarrow$

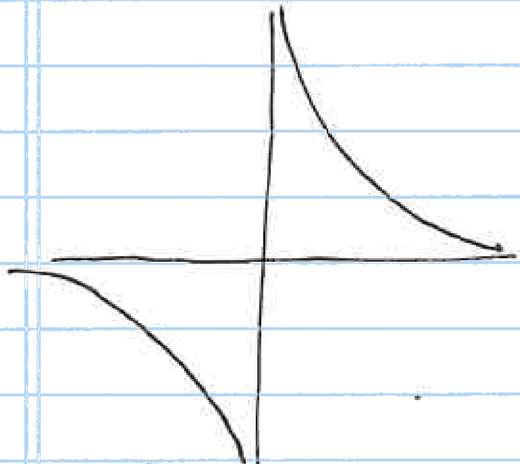
[a is an isolated pt of the domain $\underline{X}$ iff

$\exists\, r > 0$, $\{x \mid |x-a| < r\} \cap \underline{X} = \{a\}$]

This definition differs from Calculus I def^n.

Ex (a) $f(x) : \mathbb{R} - \{0\} \longrightarrow \mathbb{R} - \{0\}$

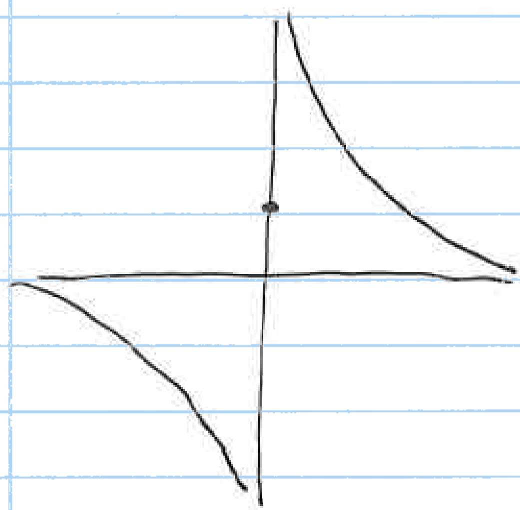
$$f(x) = \frac{1}{x}$$



Since $0 \notin$ Domain, continuity at 0 is not a part of our discussion.

f is continuous on its domain $= \mathbb{R} - \{0\}$.

(b) $g(x) = \begin{cases} 1/x & x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} : \underset{\substack{\Downarrow \\ 0}}{\mathbb{R}} \longrightarrow \mathbb{R}$



Not continuous on $\mathbb{R}$

$0 \in \mathbb{R} =$ domain

$\underbrace{\lim_{x \to 0} \frac{1}{x}}_{\text{DNE}} \neq f(0) = 1$

Not continuous at $\{0\}$

Exc # 46 p 115 Find c s.t. g is continuous

$$g(x,y) = \begin{cases} \dfrac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ c & \text{if } x = y = 0 \end{cases}$$

Away from $(0,0)$ g is continuous since $x^2+y^2 > 0$

Continuity at $(0,0)$:

Need
$$\lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2+y^2} = c$$

$$\lim_{(x,y)\to(0,0)} \underbrace{\frac{x^3}{x^2+y^2}}_{\downarrow \; 0} + \underbrace{\frac{xy^2}{x^2+y^2}}_{\downarrow \; 0} + \underbrace{\frac{2x^2+2y^2}{x^2+y^2}}_{\downarrow \; 2} = 2$$

Done it yesterday

By squeeze Thm.

similarly

$$0 \le y^2 \le x^2 + y^2$$

$$0 \le \frac{y^2}{x^2+y^2} \le 1$$

$$0 \le \left| \frac{xy^2}{x^2+y^2} \right| \le |x| \to 0$$

Take $c = 2$.

(2.2) Open + Closed sets in $\mathbb{R}^n$.

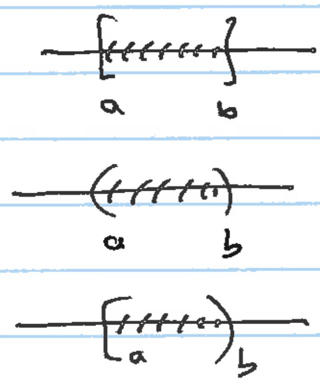
$n=1$

$[a,b] = \{x \mid a \le x \le b\}$ closed interval

$[a,b) = \{x \mid a \le x < b\}$

$(a,b) = \{x \mid a < x < b\}$ open interval

All have the same Boundary $\{a,b\}$



$\{a,b\} \subseteq [a,b]$

$\{a,b\} \cap (a,b) = \emptyset$.

<u>Defn</u> Let $S \subseteq \mathbb{R}^n$.

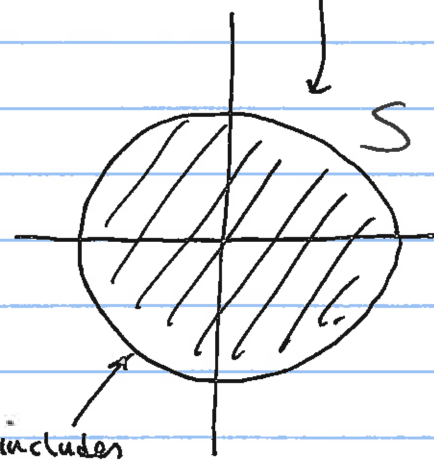
S is closed if $(\text{boundary of } S) \subseteq S$

S is open if $(\text{boundary of } S) \cap S = \emptyset$

<u>Defn</u> A point $a \in \mathbb{R}^n$ is called a boundary point of a set S if

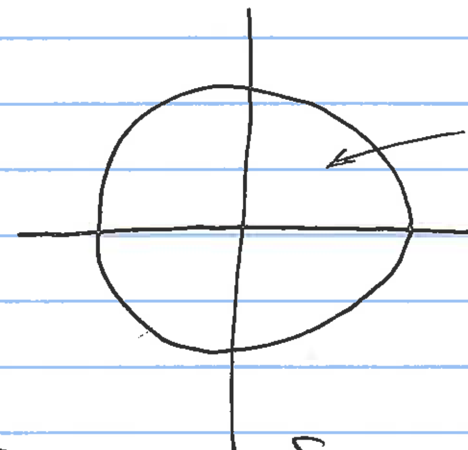
$\forall r > 0$ the set $\{x \in \mathbb{R}^n \mid \|x-a\| < r\}$ contains points of S as well as points outside S $(\mathbb{R}^n - S)$.

Ex 1 $S = \{(x,y) \mid x^2+y^2 \leq 1\}$



includes the edge circle

Boundary of $S = \{(x,y) \mid x^2+y^2 = 1\}$



empty inside

Boundary of $S \subseteq S \implies S$ is closed.

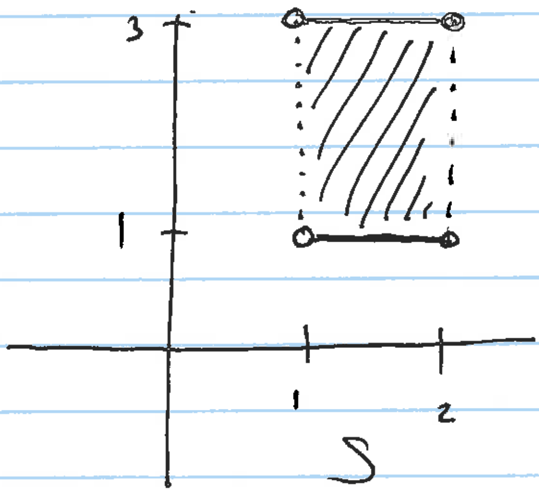
$(\text{Boundary } S) \cap S \neq \phi \implies S$ is not open.

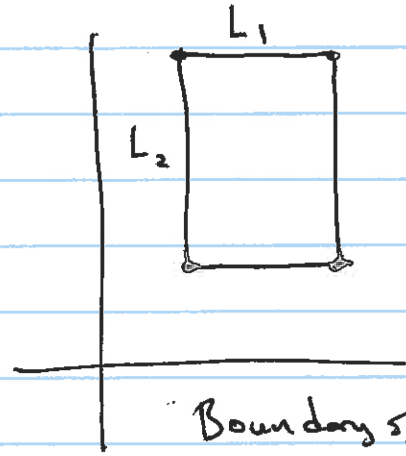
Ex 2

ϕ, $\mathbb{R}^n$ are the only sets in $\mathbb{R}^n$, which are both open and closed, since they have ϕ boundary.

Ex 3

$$S = \left\{ (x,y) \;\middle|\; \begin{matrix} 1 < x < 2 \\ 1 \le y \le 3 \end{matrix} \right\}.$$





- Boundary of S = $\{(x,y) \mid (1 \le x \le 2 \;\&\; y = 1);$ (union of 4 segments)

  OR

  $(1 \le x \le 2 \;\&\; y = 3)$

  OR

  $(x = 1 \;\&\; 1 \le y \le 3)$

  OR

  $(x = 2 \;\&\; 1 \le y \le 3)\}$

- Open? No $\left.\begin{matrix} L_1 \subseteq \text{Bd } S \\ L_1 \subseteq S \end{matrix}\right\}$ $\text{Bd } S \cap S \ne \emptyset$.

- Closed? No $\left.\begin{matrix} L_2 \subseteq \text{Bd } S \\ L_2 \not\subseteq S \end{matrix}\right\}$ $\text{Bd } S \not\subseteq S$.

Interior $S = \{(x,y) \mid 1 < x < 2, \; 1 < y < 3\}$

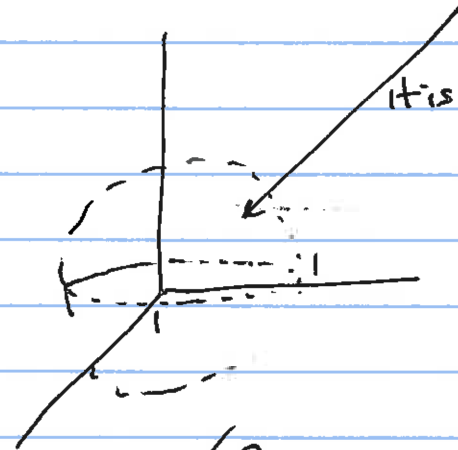
Closure $S = \{(x,y) \mid 1 \le x \le 2, \; 1 \le y \le 3\}$.

Ex 4

$D = \{(x,y,z) \mid x^2+y^2+z^2 < 1\}$

$\neq$.

unit ball, unit disc,



it is inside the sphere

$S = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

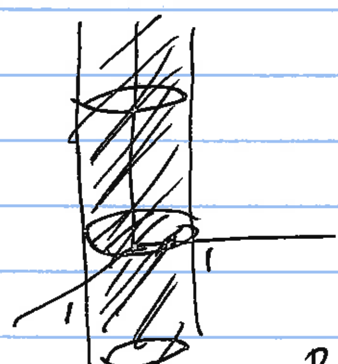
$(\text{Boundary of } D) = \underset{\text{sphere}}{S}$

$D \cap S = \emptyset$

D is open, actually we call it unit open ball

$D \cup S = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$ is called a closed ball.

Ex 5) $Z = \{(x,y,z) \mid x^2+y^2 \leq 1\}$



Solid cylinder: Z

- not open
- closed

Boundary of $Z = \{(x,y,z) \mid x^2+y^2=1\}$