Feb 9

1

Ex $\lim_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2} = ?$

Caution checking by using lines thru (0,0) will NOT be sufficient.

Solution I

$0 \le x^2 \le x^2+y^2$

$0 \le \dfrac{x^2}{x^2+y^2} \le 1$ if $(x,y) \ne (0,0)$

$-x \le x \cdot \dfrac{x^2}{x^2+y^2} \le x$

$0 \le \left| \dfrac{x^3}{x^2+y^2} \right| \le |x|.$

As $(x,y) \to (0,0)$, $x \to 0$

$0 \le \left| \dfrac{x^3}{x^2+y^2} \right| \le |x|$

we know $|x| \to 0$

Squeeze Thm. $\Rightarrow$ $\left| \dfrac{x^3}{x^2+y^2} \right| \to 0$

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0}$$

Solution

I $\varepsilon-\delta$ justification

$$0 \le x^2 \le x^2+y^2$$

$$0 \le \frac{x^2}{x^2+y^2} \le 1$$

$$\left|\frac{x^3}{x^2+y^2}\right| = |x|\left|\frac{x^2}{x^2+y^2}\right| \le |x|$$

$$\forall \varepsilon>0 \quad \exists \ \delta = \varepsilon$$

if $|(x,y)| < \delta$ $\qquad \sqrt{x^2+y^2} < \delta$

$|x| < \delta$

$$\Bigg|\underbrace{\frac{x^3}{x^2+y^2}}_{f} - \underbrace{0}_{L}\Bigg| = \left|\frac{x^3}{x^2+y^2}\right| \le |x| < \delta = \varepsilon .$$

Exercises

2.2 #7

$$\lim_{(x,y,z)\to(0,0,0)} x^2+2xy+yz+z^3+2 = 2$$

#10

$$\lim_{(x,y)\to(0,0)} \frac{e^x e^y}{x+y+2} = \frac{1}{2}.$$

#8

$$\lim_{(x,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}} \quad \underline{\underline{\text{DNE}}}$$

Along x-axis $\displaystyle \lim_{(x,\underset{\underset{y}{\parallel}}{0})\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}} = \lim_{x\to 0}\frac{0}{\sqrt{x^2}} = 0$

$$\neq$$

Along y-axis $\displaystyle \lim_{(0,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}} = \lim_{y\to 0}\frac{|y|}{\sqrt{y^2}} = \lim_{y\to 0} 1 = 1$

#13

$$\lim_{(x,y)\to(0,0)} \frac{\overbrace{x^2+2xy+y^2}^{(x+y)^2}}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x+y}$$

$$= \lim_{(x,y)\to(0,0)} x+y = 0.$$

Compare

Another Example:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+3xy+y^2}{x+y} = \underbrace{\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y}}_{\downarrow\, 0} + \underbrace{\lim \frac{xy}{x+y}}_{\text{DNE}}$$

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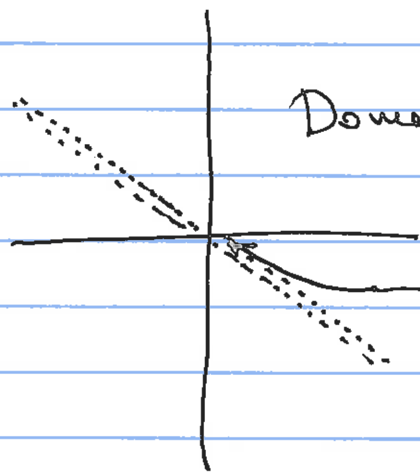
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y} \quad \underline{\underline{\text{DNE}}} \quad \text{since:}$$

not useful

$x^2 \le x^2 + y^2$ yes $\forall x, y$

$x \le x + y$ $\underline{\underline{\text{not}}}$; y can be negative



Domain of $\frac{xy}{x+y} = \{(x,y) \mid x \neq -y\}$

$y = x^3 - x$ (or something else)

Approach $(0,0)$ along $(x, x^3 - x)$

$$\frac{xy}{x+y} = \frac{(x^3 - x)\cdot x}{x^3}$$

$$= \frac{x^4 - x^2}{x^3}$$

$$= x - \frac{1}{x}.$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y} \text{ DNE} \Longleftarrow \lim_{x\to 0} x - \frac{1}{x} \text{ DNE}.$$

Next: Suppose $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + 3xy + y^2}{x+y}$ existed.

Then

$$\underbrace{\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}}_{\text{LHS DNE}} = \lim_{(x,y)\to(0,0)} \frac{x^2 + 3xy + y^2}{x+y} - \frac{x^2 + 2xy + y^2}{x+y}$$

RHS would exist

Contradiction. Hence, $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + 3xy + y^2}{x+y}$ DNE.