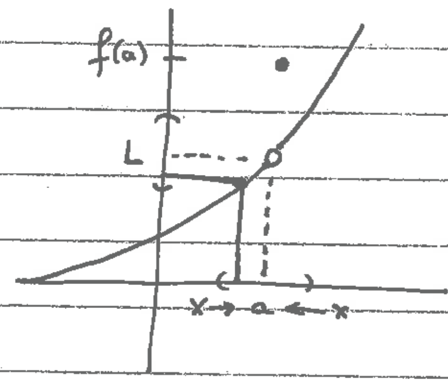
Feb 8, 2017 ①

2.2

Review $f: A \subseteq \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

$$\lim_{x \to a} f(x) = L$$



Informally Given an error margin $\varepsilon > 0$

One needs to find a small vicinity of a, $(a-\delta, a+\delta)$, s.t. $\left.\begin{array}{l} x \in (a-\delta, a+\delta) \\ \text{and} \\ x \neq a \end{array}\right\}$

will yield $f(x)$ within ε margin of error about L.

Same definition works for $f: \underline{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

Defn Let $f: \underline{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $a \in \mathbb{R}^n$

$$\lim_{x \to a} f(x) = L \in \mathbb{R}^m$$

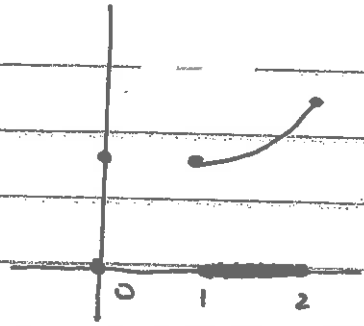
$\Longleftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in \underline{X}$

$$0 < \|x - a\| < \delta \implies \|f(x) - L\| < \varepsilon.$$

Caution! In order to have $\lim_{x \to a} f(x)$ make

Sense, one needs to have x's satisfying $x \in \underline{X}$ and $0 < \|x - a\| < \delta$ for all δ.

Ex

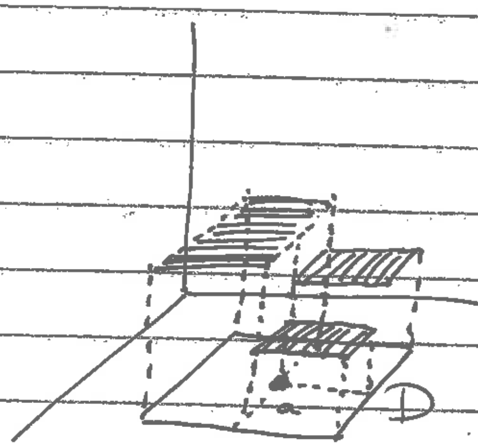


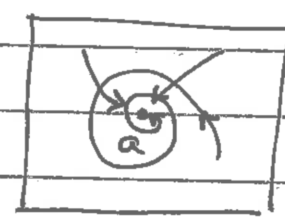
$D = \text{domain of } f = \{0\} \cup [1,2]$

$\lim_{x\to 0} f(x)$ has no meaning

since x cannot approach to 0 ($x \not\to 0$) in D.

<u>Caution 2</u> In $\mathbb{R}^n$, $n>1$ left and/or right limits do not make sense:





$D \subseteq \mathbb{R}^2$

There are many ways, directions, to approach a.

~~Ex~~ Show that

$$\lim_{(x,y)\to(1,-2)} 3x + 4y + 10 = 5$$

SOLUTION :

Want: Given $\varepsilon > 0$, I will choose $\delta = \frac{\varepsilon}{7}$ so that

If $0 < \|(x,y) - (1,-2)\| < \delta$ $\left(\begin{array}{l} \\ (x,y) \in \text{Domain} = \mathbb{R}\end{array}\right)$ then $\|f(x,y) - 5\| < \varepsilon$.

Let's show how this works: $\forall \varepsilon > 0 \ \exists \delta = \frac{\varepsilon}{7}$:

Start with $\|(x,y) - (1,-2)\| < \delta$, $(x,y) \in$ Domain.

$$\|(x-1, y+2)\| < \delta$$

$$\sqrt{(x-1)^2 + (y+2)^2} < \delta$$

$$(x-1)^2 + (y+2)^2 < \delta^2$$

$$(x-1)^2 < \delta^2 \quad \text{and} \quad (y+2)^2 < \delta^2$$

(*) $$|x-1| < \delta \quad \text{and} \quad |y+2| < \delta$$

$$\| f(x,y) - 5 \| = \| \underbrace{3x+4y+10}_{f(x,y)} - 5 \|$$

$$= \| 3x+4y+5 \|$$

$$= | 3(x-1) + \cancel{3} + 4(y+2) - \cancel{8} + \cancel{5} \|$$

$$= | 3(x-1) + 4(y+2) |$$

$$\leq |3(x-1)| + |4(y+2)|$$

$$= 3|x-1| + 4|y+2|$$

By ⊛ above

$$< 3\delta + 4\delta = 7\delta = \varepsilon \quad . \#$$

THM Let $\begin{cases} a \in \mathbb{R}^n \\ F, G : \bar{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m \\ f, g : \bar{X} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^1 \end{cases}$ be s.t.

$$\lim_{x \to a} F(\vec{x}) = L \in \mathbb{R}^m ; \quad \lim_{x \to a} f(x) = c \in \mathbb{R}$$

$$\lim_{x \to a} G(x) = M \in \mathbb{R}^m ; \quad \lim_{x \to a} g(x) = d \in \mathbb{R}.$$

Then:

$$\lim_{x \to a} F(x) + G(x) = L + M \in \mathbb{R}^m$$

$$\lim_{x \to a} f(x) F(x) = cL \in \mathbb{R}^m$$

$\forall k \in \mathbb{R}$ $$\lim_{x \to a} k \cdot F(x) = k \cdot L \in \mathbb{R}^m$$

$$\lim_{x \to a} f(x) g(x) = cd \in \mathbb{R}^1$$

• dot product $$\lim_{x \to a} F(x) \cdot G(x) = L \cdot M \in \mathbb{R}^1$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{c}{d} \text{ if } d \neq 0, \ g(x) \neq 0 \text{ on } \bar{X}.$$

Def $\Rightarrow$ $\left.\begin{array}{l} \lim\limits_{(x,y) \to (a,b)} x = a. \\ \lim\limits_{(x,y) \to (a,b)} y = b \end{array}\right\}$ Why?

Ans. $|x-a| \leq \sqrt{(x-a)^2 + (y-b)^2} = |(x,y) - (a,b)| < \delta$

Take $\varepsilon = \delta$.

Consequences Thm:

1. For any polynomial function $P(x_1, x_2, \ldots x_n)$ (examples of poly: x^2+y^3, $xz+z^5-xy^3$)

$$\lim_{(x_1,x_2,\ldots x_n)\to(a_1,a_2,\ldots a_n)} P(x_1,x_2,\ldots x_n) = P(a_1,a_2,\ldots a_n)$$

Ex $$\lim_{(x,y)\to(5,-1)} x^2+y^3-xy^2+4 =$$

By Theorem:

(Not by ε-δ proof)

$$= 5^2+(-1)^3-5(-1)^2+4$$
$$= 25-1-5+4$$
$$= 23$$

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2. $$\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{f(a)}{g(a)}$$ provided that $g(a)\neq 0$

where each f, g are polynomials:

Ex

$$\lim_{(x,y)\to(1,3)} \frac{x^2+y^2-xy}{x+y-1} = \frac{1^2+3^2-3}{\underbrace{1+3-1}_{\neq 0}} = \frac{7}{3}$$

Thm: $F: X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $F = (F_1, F_2, \dots F_m)$

3 $\lim_{x \to a} F(x) = (L_1, L_2 \dots L_m) \iff$ each i $\lim_{x \to a} F_i = L_i$ $i = 1, 2, \dots m$

Ex $$\lim_{(x,y)\to(1,3)} \left(\frac{xy}{x-y}, e^{xy}, \sin \frac{x-y}{x^2+y^2} \right)$$

$$= \left(\frac{3}{-2}, e^3, \sin \frac{-2}{10} \right)$$

Informally: calculate the limit of each component.

Ex

$$\lim_{x\to 0} \left(\frac{1}{x}, x^2 \right) = \text{DNE}$$

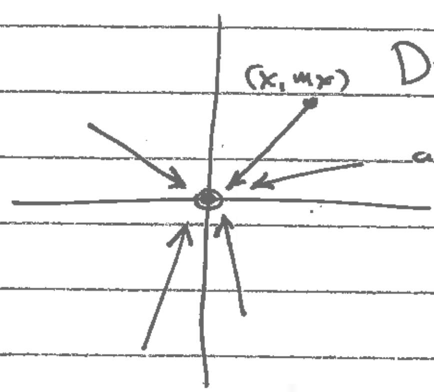
Do $\underline{\underline{\text{NOT}}}$ write (DNE, 0)

Ex 1 Define

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$$\lim_{(x,y)\to(3,4)} f(x,y) = \frac{12}{9+16} = \frac{12}{25}$$

$f(0,0)=0$; But $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ <u>Doesn't exist</u>:



Domain of $f = \mathbb{R}^2$

approach $(0,0)$ along the lines $y = mx$.

$(x, mx) \longrightarrow (0,0)$ m fixed

$x \neq 0$, $x \to 0$

$$f(x, mx) = \frac{x \cdot mx}{x^2 + (mx)^2} = \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}.$$

If $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ existed, then all of these must be equal to each other

If $m=1$, approach $(0,0)$ along (x,x): $\lim\limits_{x\to 0} f(x,x) = \frac{1}{2}$.

if $m=0$, approach $(0,0)$ along $(x,0)$, $\lim\limits_{x\to 0} f(x,0) = 0$

if $m=-1$ approach $(0,0)$ along $(x,-x)$, $\lim\limits_{x\to 0} f(x,-x) = \frac{-1}{2}$.

So $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ can't equal to any real number.