Jan 25

(1)

1.3 Continue

Given a vector $\vec{a} \neq 0$, $\dfrac{a}{\|a\|}$ is a unit vector

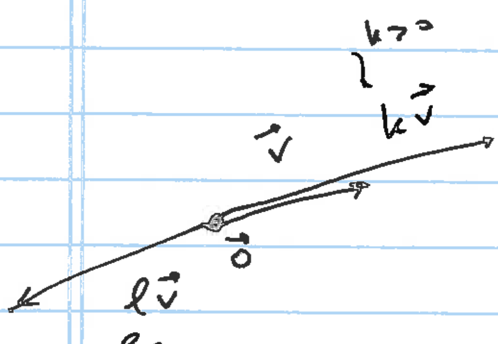
This procedure is called normalization.

Ex $5i + 6j - 4k$

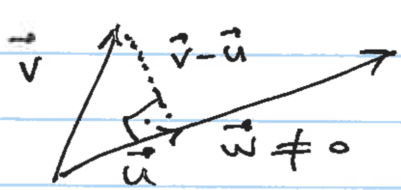
$$\|5i + 6j - 4k\| = \sqrt{25 + 36 + 16} = \sqrt{77}$$

$\dfrac{5i + 6j - 4k}{\sqrt{77}}$ is a unit vector

- parallel to $5i + 6j - 4k$
- points in the same direction



ORTHOGONAL PROJECTION:



$\vec{u} = \lambda \vec{w}$

$\vec{v} - \vec{u} \perp \vec{w}$

$\vec{u} = \mathrm{proj}_{\vec{w}} \vec{v}$

Want $\vec{v} - \vec{u} = v - \lambda \vec{w} \perp \vec{w}$

$$(\vec{v} - \lambda \vec{w}) \cdot \vec{w} = 0$$

$$\vec{v} \cdot \vec{w} - \lambda \vec{w} \cdot \vec{w} = 0$$

$$\vec{v} \cdot \vec{w} = \lambda \vec{w} \cdot \vec{w}$$

$$\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} = \lambda \qquad \text{since } \vec{w} \neq 0$$

Def $\quad \text{Proj}_{\vec{w}} \vec{v} = \dfrac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w}$

Exc # 12 $\quad \vec{a} = \vec{i} + \vec{j}$

$\quad \vec{b} = 2\vec{i} + 3\vec{j} - \vec{k}$

$$\text{proj}_a b = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a} = \frac{(2i + 3j - k) \cdot (i + j)}{(i + j) \cdot (i + j)} (i + j)$$

$$= \frac{2 \cdot 1 + 3 \cdot 1 - 1 \cdot 0}{1 \cdot 1 + 1 \cdot 1} (i + j)$$

$$= \frac{5}{2} (i + j)$$

Obs $\quad$ In general

$\text{proj}_a b \neq \text{proj}_b a$, unless $(a = b$ OR $a \cdot b = 0)$

1.4 Cross product in $\mathbb{R}^3$

Given vectors $\vec{a}$ & $\vec{b} \in \mathbb{R}^3$ we want

$\vec{a} \times \vec{b} \in \mathbb{R}^3$ s.t.

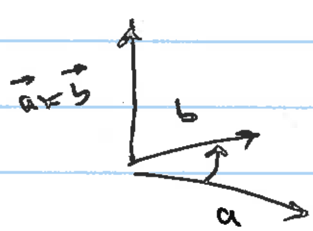
(a) $\vec{a} \times \vec{b} = 0$ if $\vec{a} = 0$ or $\vec{b} = 0$

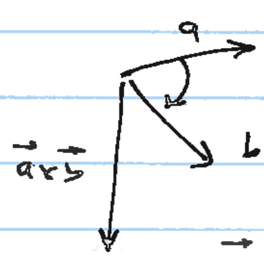
(b) if both $a, b \neq 0$, we should have

(i) $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$ where θ is the angle between $\vec{a}$ & $\vec{b}$.

(ii) $\vec{a} \times \vec{b} \perp \vec{a}$, $\vec{a} \times \vec{b} \perp \vec{b}$.

(iii) $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ satisfy Right hand Rule





Prop $(a_1, a_2, a_3) \times (b_1, b_2, b_3) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Review

Determinants:

Determinants:

1x1 $\quad \begin{vmatrix} 5 \end{vmatrix} = 5$ $\qquad |a| = a$

1x1 $\quad \begin{vmatrix} -5 \end{vmatrix} = -5$

det of 1x1 matrix $\neq$ absolute value

Notation is Very Confusing; so we avoid writing 1x1 det's.

2x2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Ex: $\begin{vmatrix} 2 & 5 \\ -1 & 6 \end{vmatrix} = 2 \cdot 6 - (-1) \cdot 5 = 17.$

3x3

Ex $\begin{vmatrix} 2 & 3 & -1 \\ 5 & 0 & 6 \\ 1 & 2 & 3 \end{vmatrix} = ?$

Method I

$$\left| \begin{array}{ccc|cc} 2 & 3 & -1 & 2 & 3 \\ 5 & 0 & 6 & 5 & 0 \\ 1 & 2 & 3 & 1 & 2 \end{array} \right.$$

$= (0 + 18 - 10) - (0 + 24 + 45) = 8 - 69 = -61.$

Method II opening along a row or column. ⑤

$$\begin{vmatrix} 2 & 3 & -1 \\ 5 & 0 & 6 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= -3\begin{vmatrix} 5 & 6 \\ 1 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 5 & 6 \end{vmatrix}$$

Signs

$$\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

$$= (-3)(9) + 0(7) - 2(17)$$
$$= -27 + 0 - 34$$
$$= -61$$

Ex). Cross product

$$(2,5,-1)\times(1,0,6) = \begin{vmatrix} i & j & k \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{vmatrix}$$

$$= +i\begin{vmatrix} 5 & -1 \\ 0 & 6 \end{vmatrix} - j\begin{vmatrix} 2 & -1 \\ 1 & 6 \end{vmatrix} + k\begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix}$$

$$= 30i - 13j - 5k.$$

Check $(30,-13,-5)\cdot(2,5,-1) = 60 - 65 + 5 = 0$

$(30,-13,-5)\cdot(1,0,6) = 30 + 0 - 30 = 0$

recall : $\left.\begin{array}{l} a\times b \perp a \\ a\times b \perp b \end{array}\right\} \Longrightarrow \begin{array}{l} (a\times b)\cdot a = 0 \\ (a\times b)\cdot b = 0 \end{array}$

Ex 2 Find all unit vectors perpendicular to both $(1,2,0)$ & $(0,1,3)$

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = (6,-3,1) = (1,2,0)\times(0,1,3)$$

$$\|(6,-3,1)\| = \sqrt{36+9+1} = \sqrt{46}$$

Ans: $\pm \frac{1}{\sqrt{46}}(6,-3,1)$

Question: Which of the following are correct

a, b

1) $a\times b = b\times a$ FALSE in general

2) $a\times(b+c) = a\times b + a\times c$ TRUE in general

$k\in\mathbb{R}$ 3) $k(a\times b) = (ka)\times b$ TRUE in general

4) $(a\times b)\times c = a\times(b\times c)$ FALSE in general

1) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ correct ($\Rightarrow$ $\vec{a}\times\vec{a} = 0$)

4) Counter example

$i\times j = k$

$i\times i = 0$

$i\times k = -j$

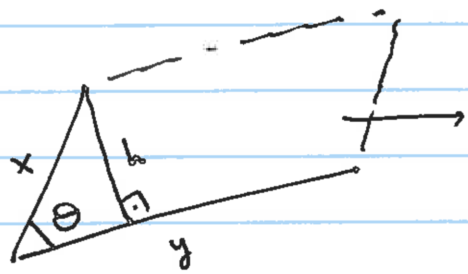
$(i\times i)\times j = 0\times j = 0$

$i\times(i\times j) = i\times k = -j$ $\neq$

Recall $a \times b \perp a, b$

$\|a \times b\| = \|a\|\|b\| \sin\theta$

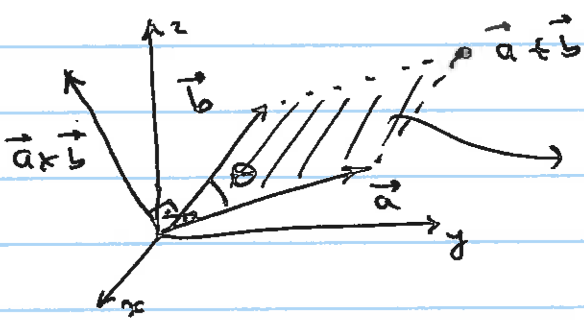
Recall HS geometry



Parallelogram area $= hy = x\sin\theta\, y$

$= xy\sin\theta$

3-D



area of this parallogram is $\|a\|\|b\|\sin\theta =$

$= \|a \times b\|$

Ex. Find the area of the parallogram with vertices

$(1,1,1), (3,2,-2), (0,3,5), (2,4,2)$

Soln move $(1,1,1)$ to $(0,0,0)$

& move all vertices in a parallel/rigid fashion:

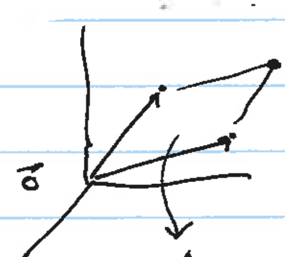
$(x,y,z) - (1,1,1)$

$(1,1,1) \to (0,0,0)$

$(3,2,-2) \to (2,1,-3)$

$(0,3,5) \to (-1,2,4)$

$(2,4,2) \to (1,3,1)$



$(1,3,1) = (2,1,-3)$
$+$
$(-1,2,4)$

Area $= \|(2,1,-3)\times(-1,2,4)\|$

$$\text{Area} = \|(2,1,-3)\times(-1,2,4)\|$$

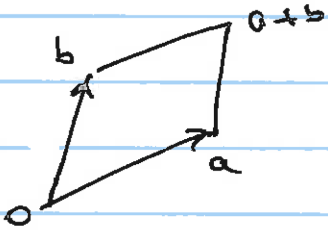
$$\begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ -1 & 2 & 4 \end{vmatrix} = (10,-5,5)$$

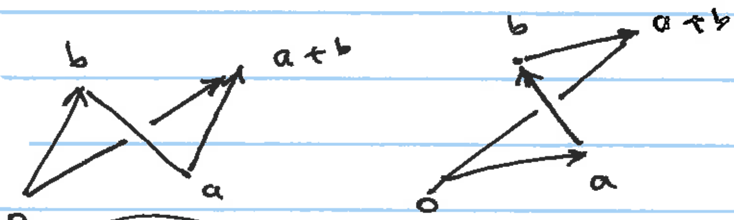
$$\|(10,-5,5)\| = \sqrt{100+25+25} = \sqrt{150}$$

Area of the parallogram with vertices $(1,1,1), (3,2,-2), (0,3,5), (2,4,2)$ is $\sqrt{150}$.

Caution:

parallelogram





not parallelograms

$$(a+b)\times a = \underbrace{a\times a}_{0} + b\times a = b\times a$$

$$(a+b)\times b = a\times b + \underbrace{b\times b}_{0} = a\times b$$

$$a\times b \qquad\qquad = a\times b$$

But, all have the same length.