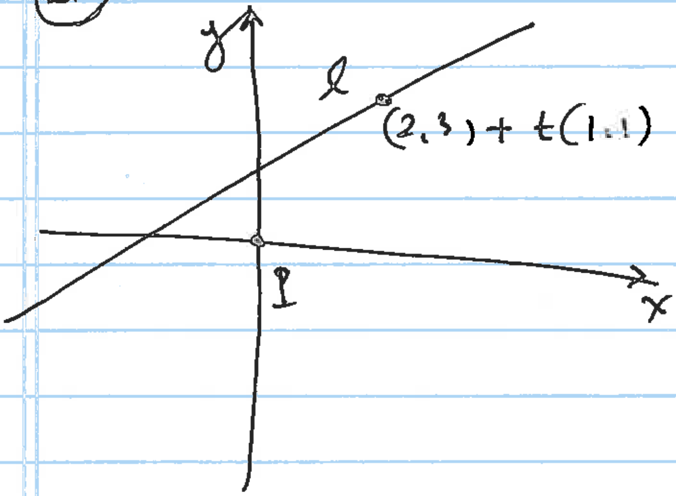
Jan 25, 2017

1.2 More examples

Ex



Given line ℓ
$r(t) = (2,3) + t(1,1)$
Find the closest pt of ℓ to $(0,0)$.

Sol'n $\quad r(t) = (2+t, 3+t)$

$$f(t) = \| r(t) - (0,0) \|^2, \quad \underline{\underline{\text{minimize}}}$$

$$= \| (2+t, 3+t) - (0,0) \|^2$$

$$= (2+t)^2 + (3+t)^2 = 4 + 4t + t^2 + 9 + 6t + t^2$$

$$= 2t^2 + 10t + 13.$$

$$f' = 4t + 10 \longrightarrow t = \frac{-10}{4} = \frac{-5}{2}$$

$$f'' = 4 \quad > 0 \qquad \text{local (absolute min)}$$

$$f\left(\frac{-5}{2}\right) = 2\left(\frac{-5}{2}\right)^2 + 10 \cdot \frac{-5}{2} + 13$$

$$= \frac{25}{2} - 25 + 13 = -\frac{25}{2} + 13 = \frac{1}{2} \qquad \text{Minimum } \underline{\underline{\text{dist}^2}}$$

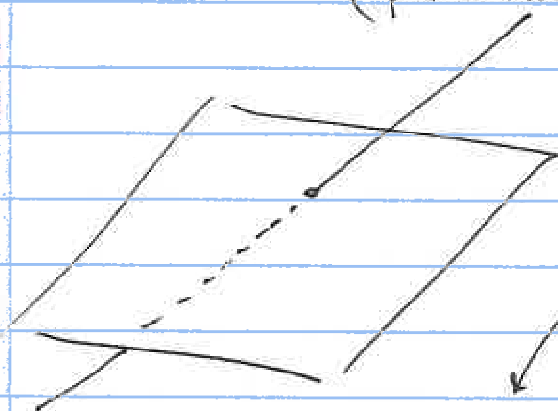
$$\text{dist}(P, \ell) = \frac{1}{\sqrt{2}}$$

* Closest pt to $(0,0)$ is : $r\left(-\frac{5}{2}\right) = \left(2 - \frac{5}{2}, 3 - \frac{5}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right)$

Exc #34 p17 Find the pt of intersection

$$l : \begin{cases} x = 1 - 4t \\ y = t - \frac{3}{2} \\ z = 2t + 1 \end{cases} \qquad \text{plane} \quad 5x - 2y + z = 1$$

(parametric) (closed)



$$5x - 2y + z = 1 \Rightarrow 5(1 - 4t) - 2\left(t - \frac{3}{2}\right) + (2t + 1)$$

$$\cancel{1} = 5 - 20t \cancel{-2t} + 3 \cancel{+2t} + \cancel{1}$$

$$20t = 8$$

$$t = \frac{8}{20} = \frac{2}{5}$$

$$x = 1 - 4t = 1 - 4 \cdot \frac{2}{5} = -\frac{3}{5}$$

$$y = t - \frac{3}{2} = \frac{2}{5} - \frac{3}{2} = \frac{-11}{10}$$

$$z = 2t + 1 = 2 \cdot \frac{2}{5} + 1 = \frac{9}{5}$$

Pt of intersection: $\left(-\frac{3}{5}, \frac{-11}{10}, \frac{9}{5}\right)$

Symmetric Equations of lines:

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$$

$b_1, b_2, b_3 \in \mathbb{R}$
$a_1, a_2, a_3 \in \mathbb{R}$
all $b_1, b_2, b_3 \neq 0$

This line passes thru: (a_1, a_2, a_3)

& line is parallel to (b_1, b_2, b_3)

Ex Given symmetric eq^n

$$\frac{x-5}{3} = \frac{y+1}{2} = \frac{z-0}{-1}$$

Find a parametric representation.

$$t = \frac{x-5}{3} = \frac{y+1}{2} = \frac{z}{-1}$$

$x-5 = 3t$ $\quad$ $x = 3t+5$
$y+1 = 2t$ $\quad$ $y = 2t-1$
$z = -t$ $\quad$ $z = -t$

$$(x,y,z) = (3t+5,\ 2t-1,\ -t)$$
$$= \underbrace{(5,-1,0)}_{\text{thru}} + t\,\underbrace{(3,2,-1)}_{\text{parallel}}$$

1.3

Defn Let $\underbrace{(a_1, a_2, \dots a_n)}_{\vec{a}}, \underbrace{(b_1, b_2, \dots b_n)}_{\vec{b}} \in \mathbb{R}^n$

$$\vec{a} \cdot \vec{b} = (a_1, a_2 \dots a_n) \cdot (b_1, b_2, \dots b_n)$$

Called Dot product

$$= a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n$$

Ex

i) $(2, 3, -1) \cdot (6, 0, 5) = 12 + 0 - 5 = 7.$

ii)
$$(\vec{i} + 5\vec{j} + \vec{k}) \cdot (3i - 7\vec{k})$$

$$= 1 \cdot 3 + 5 \cdot 0 + 1 \cdot (-7)$$
$$= 3 - 7 = -4.$$

Prop 1) $\vec{a} \cdot \vec{a} = (a_1, a_2, \dots a_n) \cdot (a_1, \dots a_n)$
$$= a_1^2 + a_2^2 + \dots + a_n^2 \geqslant 0$$

2) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

3) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

4) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

5) If $k \in \mathbb{R}$ $\quad (k \cdot \vec{a}) \cdot \vec{b} = k \cdot (\vec{a} \cdot \vec{b})$

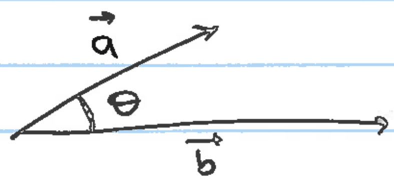
Prop Let $\vec{a}, \vec{b}$ be vectors in $\mathbb{R}^n$

$\vec{a} \neq 0$, $\vec{b} \neq 0$.

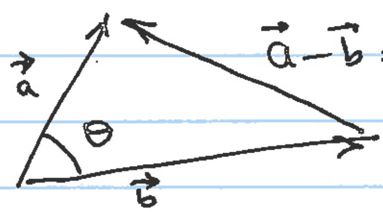
Then

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$$

where θ is the angle between the vectors $\vec{a}$ & $\vec{b}$.



Proof:



Law of Cosines:

$$\|\vec{c}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|a\|\|b\|\cos\theta.$$

$$\| $$

$$\vec{c} \cdot \vec{c}$$

$$\| $$

$$(\vec{a}-\vec{b}) \cdot (\vec{a}-\vec{b})$$

$$\| $$

$$\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$$

$$\| $$

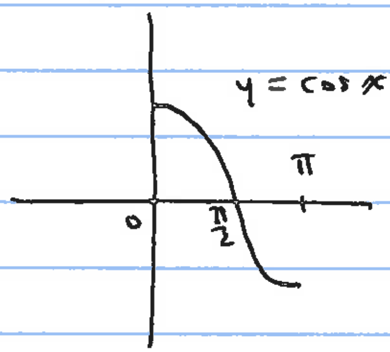
$$\|a\|^2 - 2\vec{a} \cdot \vec{b} + \|b\|^2.$$

$$\Longrightarrow \quad 2\vec{a} \cdot \vec{b} = 2\|a\|\|b\|\cos\theta.$$

#

Consequences of $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$.

i) $\theta = \cos^{-1}\dfrac{a\cdot b}{\|a\|\,\|b\|}$ if $\vec{a}\neq 0$, $\vec{b}\neq 0$



$\cos x : [0,\pi] \longrightarrow [-1,+1]$.

$\cos^{-1} : [-1,1] \longrightarrow [0,\pi]$.

Cauchy-Schwartz Inequality

ii) $|a\cdot b| \leq \|a\|\,\|b\|$ since $|\cos\theta| \leq 1$

iii) $\vec{a}\neq 0$, $\vec{b}\neq 0$

a is perpendicular to b, i.e. $\theta = \pi/2$: $\vec{a}\perp\vec{b} \iff \vec{a}\cdot\vec{b} = 0$

Ex 1.3 Exc # 8 p 26

Find the angle between $\underbrace{(-1,2)}_{a}$ & $\underbrace{(3,1)}_{b}$

$\|a\| = \|(-1,2)\| = \sqrt{5}$

$\|b\| = \|(3,1)\| = \sqrt{10}$

$a\cdot b = (-1,2)\cdot(3,1) = -3+2 = -1$

$$\theta = \cos^{-1}\left(\frac{a\cdot b}{\|a\|\,\|b\|}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{10}}\right) = \cos^{-1}\frac{-1}{5\sqrt{2}}$$

$\pi/2 < \theta < \pi$, θ is obtuse.